APPROXIMATE CONTROLLABILITY OF IMPULSIVE FRACTIONAL STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY AND POISSON JUMPS

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ABSTRACT. The main purpose of this paper is to investigate the result on approximate controllability of stochastic integro-differential equations with state-dependent delay and Poisson jumps. By employing the stochastic analysis theory and fixed point theorem, a set of novel sufficient conditions are derived for the second order nonlinear impulsive fractional neutral stochastic integro-differential systems with state-dependent delay and Poisson jumps. Finally, an example has been given to validate the efficiency of the proposed theoretical results.

1. INTRODUCTION

In recent few decades, the theory of fractional calculus has become a most interesting area for researchers due to its wide applicability in sciences and engineering such as material sciences, mechanics, seepage flow in porous media, in fluid dynamic traffic models, population dynamics, economics, chemical technology, medicine and many others. One of the major applications of fractional calculus is the hypothesis of fractional evolution equations. In fact, fractional differential equations may be considered as an alternative model to nonlinear partial differential equations. The nonlinear oscillations of an earthquake can be described by the fractional differential equation. The fractional derivatives give a phenomenal instrument for describing the memory and genetic properties of different materials and process which is a major advantage of fractional calculus. For more details about fractional calculus and fractional differential equations, we refer to the monographs [32, 26, 38, 22] and references cited therein.

To build more realistic models in chemistry, physics, economics, social sciences, finance and other areas, stochastic effects need to be taken into account. Therefore, many real world problems can be modeled by stochastic differential equations. The existence, stability, approximate controllability, uniqueness of solutions of the stochastic differential equation with delay is a special type of stochastic functional differential equations have recently received a lot of attentions (see [30, 29, 15, 16, 34, 33, 13, 18, 31, 14, 23]).

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problem. The leading deal with nonlocal conditions due to Byszewski [11, 12]. Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problem. Stochastic differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained. For more details about nonlocal conditions (see [10, 30, 29, 21, 4]).

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to [6, 8, 25]. However, impulsive differential equations of fractional order have not been much studied and many aspects of these equations are yet to be explored. For some recent work on impulsive fractional differential equations, see [2, 3, 9, 7, 45, 40, 41] and the references therein.

The controllability is one of the basic ideas in linear and nonlinear control theory, and plays a crucial role in both deterministic and stochastic control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Many papers have been dedicated to the approximate controllability of fractional differential equations for example [46, 34, 29, 30, 13, 16, 28, 33].

The Poisson jumps have become very popular, it is extensively used to model many of the phenomena arising in areas such as economic, finanace, physics, biology, medicine and other sciences. For example if a system jumps from a "normal state" to a "bad state", the strength of system in random. It is natural and necessary to include a jump term in the stochastic differential equation. Sakthivel and Ren [37] established the exponential stability of second-order stochastic evolution equations with Poisson jumps. Existence and uniqueness of solutions to neutral stochastic functional differential equations with Poisson jumps are derived by Tan et.al [44]. Moreover, stochastic differential equations with Poisson jumps are analyzed by many authors [29, 24, 37, 5]. In particluar, Muthukumar and Rajivgandhi [30] established the approximate controllability of fractional order stochastic variational inequalities driven by Poisson jumps. Recently, Sathiyaraj et.al [39] derived fractional order stochastic dynamical systems with distributed delayed control and Poisson jumps. Very recently Muthukumar and Thiagu [29] studied existence of solutions and approximate controllability of fractional nonlocal neutral impulsive stochastic differential equations of order $1 < q < 2$ with infinite delay and Poisson jumps. However, approximate controllability of nonlocal fractional neutral impulsive stochastic integro-differential equations of order $1 < q < 2$ with state-dependent

delay and Poisson jumps have not yet been fully considered in the literature, and this fact is the main motivation of this work.

We consider the following neutral integro-differential equations of fractional-order with state-dependent delay and Poisson jumps of the model:

$$
{}^{C}D_t^q\left[y(t) + \mathcal{A}_1\left(t, y_t, \int_0^t a(t, s, y_s)ds\right)\right] = [\mathscr{A}y(t) + Bu(t)]dt + \int_0^t \mathcal{A}_2(t-s)y(s)ds
$$

$$
+ \int_{-\infty}^t h(\tau, y_{\varrho(\tau, y_\tau)})dW(\tau) + \int_Z g(t, y_t, z)\widetilde{N}(dt, dz), t \in \mathscr{I} := [0, T]\backslash\{t_1, \cdots, t_n\},
$$

$$
\tag{1.1}
$$

$$
y_0(t) = \varphi(t) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(t), \quad t \in (-\infty, 0], \tag{1.2}
$$

$$
y'(0) = \xi, \tag{1.3}
$$

$$
\Delta y(t_p) = \mathcal{I}_p(y_{t_p}), \tag{1.4}
$$

$$
\Delta y'(t_p) = \overline{\mathcal{I}}_p(y_{t_p}), \quad p = 1, 2, \cdots, n. \tag{1.5}
$$

Here, the state variable $y(\cdot)$ takes values in a real separable Hilbert spaces $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_{\mathcal{H}}$, ${}^{C}D_t^q$ denotes the Caputo derivative of order q, where $1 < q < 2$. $\mathscr{A}, (\mathcal{A}_2(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in Hilbert space $\mathcal{H}$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$ be the given time points. The control function $u(\cdot)$ is given in $\mathcal{L}_2(\mathscr{I}, U)$ of admissible control functions with U as a Hilbert space. B is a bounded linear operator from U into $\mathcal{H}$. Also $\mathscr{A} : D(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on $\mathcal{H}$. Let $\mathcal{K}$ be the another separable Hilbert space. Let $\{W(t)\}_{t\geq 0}$ be a given $\mathcal{K}$-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Let $\overline{q} = \{\overline{q}(t) : t \in D_{\overline{q}}\}$ be a stationary $\mathfrak{F}_t$-Poisson point process with characteristic measure κ. Let $N(dt, dz)$ be the Poisson counting measure associated with $\overline{q}$. Then $N(t, Z) = \sum\limits_{s \in D_{\overline{q}}, s\leq t} \mathcal{I}_Z(\overline{q}(s))$ with measurable set $Z \in \overline{B}(\mathcal{K}-\{0\})$, which denotes the Borel $\sigma-$ field of $\mathcal{K} - \{0\}$. Let $\widetilde{N}(dt, dz) = N(dt, dz) - dt\kappa(dz)$ be the compensated Poisson measure that is independent of $W(t)$. Let $\mathcal{P}_2([0, T] \times Z; \mathcal{H})$ be the space all mapping $\chi : [0, T] \times Z \to \mathcal{H}$ for which $\int_0^T \int_Z \mathbb{E}\|\chi(t, z)\|_{\mathcal{H}}^2 dt\kappa(dz) < \infty$. We can define $\mathcal{H}-$ valued stochastic integral $\int_0^T \int_Z \chi(t, z)\overline{N}(dt, dz)$, which is a centred square integrable martingale. We can also employ the same notation $\| \cdot \|$ for the norm of $\mathscr{L}(\mathcal{K}, \mathcal{H})$, which denotes the space of all bounded operators from $\mathcal{K}$ into $\mathcal{H}$. Simply $\mathscr{L}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$. The histories y_t represents the function defined by $y_t : (-\infty, 0] \to \mathcal{H}, y_t(\theta) = y(t + \theta)$, for $t \leq 0$ belongs to some phase space $\mathscr{B}$ described axiomatically. Furthermore, $\mathcal{A}_1 : \mathscr{I} \times \mathscr{B} \times \mathcal{H} \to \mathcal{H}, h : \mathscr{I} \times \mathscr{B} \to \mathscr{L}_Q(\mathcal{K}, \mathcal{H}), g : \mathscr{I} \times \mathscr{B} \times Z \to \mathcal{H}$ and $\varrho : \mathscr{I} \times \mathscr{B} \to (-\infty, T]$ are nonlinear functions. Here, $\mathscr{L}_Q(\mathcal{K}, \mathcal{H})$ denotes the space of all $Q-$ Hilbert Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$. Moreover, $m_1 : \mathscr{B}^m \to \mathscr{B}$ is a continuous function. $\mathcal{I}_p$ and $\overline{\mathcal{I}}_p : \mathcal{H} \to \mathcal{H}$ are appropriate functions. The symbol $\Delta\zeta(t)$ represents the jump of the function ζ at t which is defined by $\Delta\zeta(t) = \zeta(t^+) - \zeta(t^-)$. The initial data $\varphi = \{\varphi(t) : t \in (-\infty, 0]\}$ is an $\mathfrak{F}_0-$ measurable $\mathscr{B}-$ valued stochastic process independent of Brownian mation $\{W(t)\}$ and Poisson point process $\overline{q}(\cdot)$ with finite second moment. Furthermore, $\xi(t)$ is an $\mathfrak{F}_t-$ measurable $\mathcal{H}-$ valued random variable independent of $W(t)$ and Poisson point process $\overline{q}$ with finite second moment.

The rest of this paper is organized as follows, In Section 2, we summarize several important working tools on second-order fractional derivative and we recall some preliminary results about analytic semigroups, delay definitions and its generator that will be used to develop our outcomes. In Section 3, by the Sadovskii's fixed point theorem, we consider a sufficient condition for the existence for mild solutions of model (1.1)-(1.5). In Section 4, the system (1.1)-(1.5) has approximate controllability. In Section 5, we give an example to illustrate the efficiency of the obtained results.

2. Preliminaries

Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ denote two real separable Hilbert spaces. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space furnished with a normal filtration $\{\mathfrak{F}_t, t \in \mathscr{I}\}$ satisfying the usual conditions (i.e., right continuity and $\mathfrak{F}_0$ containing all $\mathbb{P}$-null sets of $\mathfrak{F}$). An $\mathcal{H}$-valued random variable is an $\mathfrak{F}$-measurable function $y(t)$: $\Omega \to \mathcal{H}$ and the collection of random variables $\mathcal{W} = \{y(t, \omega): \Omega \to \mathcal{H}|_{t \in \mathscr{I}}\}$ is called a stochastic process. Usually, we suppress the dependence on $\omega \in \Omega$ and write $y(t)$ instead of $y(t, \omega)$ and $y(t): \mathscr{I} \to \mathcal{H}$ in the place of $\mathcal{W}$. Let $\{\epsilon_k\}_{k=1}^{\infty}$ be a complete orthonormal basis of $\mathcal{K}$. Suppose that $\{W(t); t \geq 0\}$ is a $\mathcal{K}$-valued Wiener process with finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum\limits_{k=1}^{\infty} \alpha_k < \infty$, which satisfies $Q\epsilon_k = \alpha_k \epsilon_k$. We denote $W(t) = \sum\limits_{k=1}^{\infty} \sqrt{\alpha_k}\beta_k(t)\epsilon_k$, where $\{\beta_k(t)\}_{k=1}^{\infty}$, are mutually independent one-dimensional standard Wiener processes. We assume that $\mathfrak{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$ is the $\sigma-$algebra generated by W and $\mathfrak{F}_T = \mathfrak{F}$. Let $\varsigma \in \mathscr{L}(\mathcal{K}, \mathcal{H})$ and define

$$\|\varsigma\|_Q^2 = Tr(\varsigma Q \varsigma^*) = \sum_{k=1}^{\infty} \|\sqrt{\alpha_k}\varsigma \epsilon_k\|^2.$$

If $\|\varsigma\|_Q^2 < \infty$, then ς is called a Q-Hilbert-Schmidt operator. Let $\mathscr{L}_Q(\mathcal{K}, \mathcal{H})$ denote the space of all Q-Hilbert-Schmidt operators $\varsigma : \mathcal{K} \to \mathcal{H}$. The completion $\mathscr{L}_Q(\mathcal{K}, \mathcal{H})$ of $\mathscr{L}(\mathcal{K}, \mathcal{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$, where $\|\varsigma\|_Q^2 = \langle \varsigma, \varsigma \rangle$ is a Hilbert space with the above norm topology. The collection of all strongly measurable, square integrable $\mathcal{H}$-valued random variables, denoted by $\mathscr{L}_2(\Omega, \mathfrak{F}, \mathbb{P}; \mathcal{H}) \equiv \mathscr{L}_2(\Omega; \mathcal{H})$, is a Banach space equipped with the norm $\|y(\cdot)\|_{\mathscr{L}_2} = (\mathbb{E}\|y(\cdot, \omega)\|_{\mathcal{H}}^2)^{\frac{1}{2}}$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}(h_1) = \int_{\Omega} h_1(\omega)d\mathcal{P}$. Let $\hat{\mathscr{I}} = (-\infty, T]$ and $C(\hat{\mathscr{I}}, \mathscr{L}_2(\Omega; \mathcal{H}))$ be the Banach space of all continuous maps from $\hat{\mathscr{I}}$ into $\mathscr{L}_2(\Omega; \mathcal{H})$ satisfying the condition $\sup\limits_{t \in \hat{\mathscr{I}}} \mathbb{E}\|y(t)\|^2 < \infty$.

Now, we present the abstract phase space $\mathscr{B}$. Assume that the phase space $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathcal{H}$ and fulfilling the subsequent elementary axioms as a result of Hale and kato [20].

If $y : (-\infty, T] \to \mathcal{H}, T > 0$ is continuous on $\mathscr{I}$ and $y_0 \in \mathscr{B}$, then for every $t \in \mathscr{I}$ the following conditions hold:

(A1) y_t is in $\mathscr{B}$;
(A2) $\|y(t)\| \leq H\|y_t\|_{\mathscr{B}}$;

(A3) $\|y_t\|_{\mathscr{B}} \leq K_1(t) \sup_{0 \leq s \leq t} \|y(s)\| + K_2(t)\|y_0\|_{\mathscr{B}_h}$, where $H > 0$ is a constant: $K_1(t), K_2(t) : [0, +\infty) \to [1, +\infty), K_1$ is continuous, K_2 is locally bounded, K_1 and K_2 are independent of $y(\cdot)$.

$(A4)$ The function $t \to \varphi_t$ is well described and continuous from the set

$$\mathscr{R}(\varrho^-) = \{\varrho(s, \psi) : (s, \psi) \in \mathscr{I} \times \mathscr{B}\},$$

into $\mathscr{B}$ and there is a continuous and bounded function $J^\varphi : \mathscr{R}(\varrho^-) \to (0, \infty)$ to ensure that $\mathbb{E}\|\varphi_t\|_{\mathscr{B}}^2 \leq J^\varphi(t)\mathbb{E}\|\varphi\|_{\mathscr{B}}^2$ for every $t \in \mathscr{R}(\varrho^-)$.

(A5) For the function $y(\cdot)$ in (A1), y_t is a $\mathscr{B}$ valued continuous functions on $[0, T)$.

(A6) The space $\mathscr{B}$ is complete.

Recognize the space $\mathscr{B}_T = \{y : (-\infty, T] \to \mathcal{H}$ such that $y_0 \in \mathscr{B}$ and the constraint $y|_{\mathscr{I}} \in \mathcal{PC}(\mathscr{I}, \mathscr{L}^2)\}$.

The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is denoted by

$$\|y\|_{\mathscr{B}_T} = \|\varphi\|_{\mathscr{B}} + \sup\{(\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}} : s \in [0, T]\}, \quad y \in \mathscr{B}_T.$$

Lemma 1. [17] *Let* $y: (-\infty, T] \to \mathcal{H}$ *be a function in a way that* $y_0 = \varphi$ *and* $y \in \mathcal{PC}(\mathscr{I}, \mathscr{L}^2)$ *and if* $(A4)$ *holds. Then*

$$\mathbb{E}\|y_s\|_{\mathscr{B}}^2 \leq K_1^{*2} \sup\{\mathbb{E}\|y(\theta)\|_{\mathcal{H}}^2 : \theta \in [0, \max\{0, s\}]\} + (K_2^* + J^\varphi)^2 \mathbb{E}\|y_0\|_{\mathscr{B}}^2, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $J^\varphi = \sup_{t \in \mathcal{R}(\varrho^-)} J^\varphi(t), \quad K_1^* = \sup_{t \in \mathscr{I}} K_1(t), \quad K_2^* = \sup_{t \in \mathscr{I}} K_2(t).$

To stay way from the reiterations of a few definitions utilized as a part of this paper we refer to the readers such as for the definitions of the fractional integral, Riemann-Liouville fractional operator, resolvent operator and Caputo's derivative one can see papers [21, 42, 29] and the monograps [47, 22, 32].

Consider the abstract fractional integro-differential equations as in the form

$$D_t^q y(t) = \mathscr{A}y(t) + \int_0^t \mathcal{A}_2(t - s)y(s)ds,$$

$$y(0) = y_0 \in \mathcal{H}, \quad y'(0) = 0, \tag{2.1}$$

which is an associated q- resolvent operator of bounded linear operator $(\mathcal{R}_q(t))_{t \geq 0}$ on $\mathcal{H}$.

Definition 2.1. *A one-parameter family of bounded linear operator* $(\mathcal{R}_q(t))_{t \geq 0}$ *on* $\mathcal{H}$ *is called an* q- *resolvent operator of* (2.1) *if the following conditions are verified.*

   (i) *The function* $\mathcal{R}_q(\cdot) : [0, \infty) \to \mathscr{L}(\mathcal{H})$ *is strongly continuous and* $\mathcal{R}_q(0)y = y$ *for all* $y \in \mathcal{H}$ *and* $q \in (1, 2)$.
   (ii) *For* $y \in D(\mathscr{A}), \mathcal{R}_q(\cdot)y \in C([0, \infty), [D(\mathscr{A})]) \cap C'((0, \infty), \mathcal{H})$, *and*

$$D_t^q \mathcal{R}_q(t)y = \mathscr{A}\mathcal{R}_q(t)y + \int_0^t \mathcal{A}_2(t - s)\mathcal{R}_q(s)yds$$

$$D_t^q \mathcal{R}_q(t)y = \mathcal{R}_q(t)\mathscr{A}y + \int_0^t \mathcal{R}_q(t - s)\mathcal{A}_2(s)yds$$

for every $t \geq 0$.

In order to see the existence of q- resolvent operator for problem (2.1), we have considered the following conditions

(G1) The operator $\mathscr{A} : D(\mathscr{A}) \subseteq \mathcal{H} \to \mathcal{H}$ is a closed linear operator with $[D(\mathscr{A})]$ dense in $\mathcal{H}$. Let $1 < q < 2$. For some $\widetilde{\varphi}_0 \in (0, \frac{\pi}{2}]$, for each $\widetilde{\varphi} < \widetilde{\varphi}_0$, there is a positive constant $C_0 = C_0(\widetilde{\varphi})$ such that $\kappa \in p(\mathscr{A})$ for each $\kappa \in \sum_{0,q\alpha} = \{\kappa \in \mathbb{C} : \kappa \neq 0, |\arg(\kappa)| < q\beta\}$, where $\beta = \widetilde{\varphi} + \frac{\pi}{2}$ and $\|R(\kappa, \mathscr{A})\|_{\mathcal{H}} \leq \frac{C_0}{|\kappa|}$ for all $\kappa \in \sum_{0,q\beta}$.

(G2) For all $t \geq 0, \mathcal{A}_2(t) : D(\mathcal{A}_2(t)) \subseteq \mathcal{H} \to \mathcal{H}$ is a closed linear operator, $D(\mathscr{A}) \subseteq D(\mathcal{A}_2(t))$ and $\mathcal{A}_2(\cdot)y$ is strongly measurable on $(0, \infty)$ for each $y \in D(\mathscr{A})$. There exists $b_1(\cdot) \in \mathscr{L}_{loc}^1(\mathbb{R}^+)$ such that $\bar{b}_1(\kappa)$ exists for $Re(\kappa) > 0$ and $\|\mathcal{A}_2(t)y\|_{\mathcal{H}} \leq b_1(t)\|y\|_1$, for all $t > 0$ and $y \in D(\mathscr{A})$. Moreover, the operator valued function $\overline{\mathcal{A}_2} : \sum_{0,\frac{\pi}{2}} \to \mathscr{L}([D(\mathscr{A})], \mathcal{H})$ has an analytic extension (still denoted by $\overline{\mathcal{A}_2}$) to $\sum_{0,\beta}$ such that $\|\overline{\mathcal{A}_2}(\kappa)y\|_{\mathcal{H}} \leq \|\overline{\mathcal{A}_2}(\kappa)\|_{\mathcal{H}}\|y\|_1$ for all $y \in D(\mathscr{A})$, and $\|\overline{\mathcal{A}_2}(\kappa)\| = 0\left(\frac{1}{|\kappa|}\right)$ as $|\kappa| \to \infty$.

(G3) There exists a subspace $D \subseteq D(\mathscr{A})$ dense in $[D(\mathscr{A})]$ and a positive constant $\mathcal{M}_0$ such that $\mathscr{A}(D) \subseteq D(\mathscr{A}), \overline{\mathcal{A}_2}(\kappa)(D) \subseteq D(\mathscr{A})$, and $\|\mathscr{A}\overline{\mathcal{A}_2}(\kappa)y\|_{\mathcal{H}} \leq \mathcal{M}_0\|y\|_{\mathcal{H}}$ for every $y \in D$ and all $\kappa \in \sum_{0,\beta}$.

In the sequel for $k_1 > 0$ and $\alpha \in (\frac{\pi}{2}, \beta), \sum_{k_1,\alpha} = \{\kappa \in \mathbb{C} : \kappa \neq 0, |\kappa| > k_1, |\arg(\kappa)| < \alpha\}$ for $\Phi_{k_1,\alpha}, \Phi_{k_1,\alpha}^i, i = 1, 2, 3$ are the paths $\Phi_{k_1,\alpha}^1 = \{te^{i\alpha} : t \geq k_1\}, \Phi_{k_1,\alpha}^2 = \{te^{i\mu} : |\mu| \leq \alpha\}, \Phi_{k_1,\alpha}^3 = \{te^{-i\alpha} : t \geq k_1\}$, and $\Phi_{k_1,\alpha} = \cup_{i=1}^3 \Phi_{k_1,\alpha}^i$ oriented counterclockwise. In addition $\epsilon_q(F_q)$ are the sets

$$\epsilon_q(F_q) = \left\{\kappa \in \mathbb{C} : F_q(\kappa) = \kappa^{q-1}(\kappa^q I - \mathscr{A} - \overline{\mathcal{A}_2}(\kappa))^{-1} \in \mathscr{L}(\mathcal{H})\right\}.$$

Define the operator family $(\mathcal{R}_q(t))_{t \geq 0}$ by

$$\mathcal{R}_q(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Phi_{n,\alpha}} e^{\kappa t} F_q(\kappa)d\kappa, t > 0, \\ I, t = 0. \end{cases} \tag{2.2}$$

Theorem 2.1. *Assume that conditions $(G1) - (G3)$ are fulfilled. Then there exists a unique q-resolvent operator for problem (2.1).*

Theorem 2.2. [1] *The function $\mathcal{R}_q : [0, \infty) \to \mathscr{L}(\mathcal{H})$ is strongly continuous and $\mathcal{R}_q : (0, \infty) \to \mathscr{L}(\mathcal{H})$ is uniformly continous.*

Now, we consider the non-homogeneous problem

$$D_t^q y(t) = \mathscr{A}y(t) + \int_0^t \mathcal{A}_2(t-s)y(s)ds + f(t), t \in [0, T],$$
$$y(0) = y_0, \quad y'(0) = 0, \tag{2.3}$$

where $q \in (1, 2)$ and $f \in \mathscr{L}^1([0, T], \mathcal{H})$. In the sequel, $\mathcal{R}_q(\cdot)$ is the operator function defined by (2.2).

$^cD_t^q\overline{g}(t)$ represents the Caputo derivative of order $q > 0$ of $\overline{g}$ is defined by

$$^cD_t^q\overline{g}(t) = \int_0^t h_{n-q}(t-s)\frac{d^n}{ds^n}\overline{g}(s)ds$$

where n is the smallest integer greater that or equal to q and $h\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, t > 0, \gamma \geq 0$.

Definition 2.2. [43] *A function $y : [0, T] \to \mathcal{H}, 0 < T$ is called a classical solution of (2.3) on $[0, T]$ if $y \in C([0, T], [D(\mathscr{A})]) \cap C([0, T], \mathcal{H}), h_{n-q} * y \in C^1([0, T], \mathcal{H}), n = 1, 2$, the conditions $y(0) = y_0, y'(0) = 0$ holds and (2.3) is verified on $[0, T]$.*

Definition 2.3. [43] *Let $q \in (1, 2)$, we define the family $(\mathcal{T}_q(t))_{t \geq 0}$ by*

$$\mathcal{T}_q(t)y = \int_0^t h_{q-1}(t - s)\mathcal{R}_q(s)y ds,$$

for each $t \geq 0$.

Lemma 2. [1] *The function $\mathcal{R}_q(\cdot)$ is exponentially bounded in $\mathscr{L}(\mathcal{H})$.*

Lemma 3. [1] *The function $\mathcal{R}_q(\cdot)$ is exponentially bounded in $\mathscr{L}(\mathcal{H})$, then $\mathcal{T}_q(\cdot)$ is exponentially bounded in $\mathscr{L}(\mathcal{H})$.*

Lemma 4. [1] *The function $\mathcal{R}_q(\cdot)$ is exponentially bounded in $\mathscr{L}([D(\mathscr{A})])$, then $\mathcal{T}_q(\cdot)$ is exponentially bounded in $\mathscr{L}([D(\mathscr{A})])$.*

We denote by $(-\mathscr{A})^\beta$ the fractional power of the operator $-\mathscr{A}$ for $0 \leq \beta \leq 1$, one have the following next result.

Lemma 5. [1] *Suppose that the conditions $(G1) - (G3)$ are satisfied. Let $q \in (1, 2)$ and $\beta \in (0, 1)$ such that $q\beta \in (0, 1)$, then there exists a positive number $\mathcal{M}_1$ such that*

$$\|(-\mathscr{A})^\beta \mathcal{R}_q(t)\| \leq \mathcal{M}_1 e^{rt} t^{-q\beta},$$
$$\|(-\mathscr{A})^\beta \mathcal{T}_q(t)\| \leq \mathcal{M}_1 e^{rt} t^{q(1-\beta)-1},$$

for all $t > 0$. If $y \in [D(-\mathscr{A})^\beta]$, then $(-\mathscr{A})^\beta \mathcal{R}_q(t)y = \mathcal{R}_q(t)(-\mathscr{A})^\beta y$ and $(-\mathscr{A})^\beta \mathcal{T}_q(t)y = \mathcal{T}_q(t)(-\mathscr{A})^\beta y$.

3. Main Results

In this section, we derive the existence of mild solution to systems (1.1)-(1.5) by using Sadovskii's fixed point theorem and fractional calculus.

Definition 3.1. [19, 35, 43] *An $\mathcal{H}$-valued stochastic process $\{y(t) : t \in (-\infty, T]\}$ is said to be the mild solution of the system (1.1)-(1.5) if*

(i) *$y(t)$ is a $\mathfrak{F}_t$-adapted and measurable for $t \geq 0$.*
(ii) *$y(t)$ is continuous on $\mathscr{I}$ almost surely and for each $s \in [0, t)$, the function $\mathscr{A}\mathcal{T}_q(t - s)\mathcal{A}_1\left(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\right)$ is integrable such that the following*

stochastic integral equation is satisfied

$$y(t) = \begin{cases} \mathcal{R}_q(t)\big[\varphi(0) + \mathcal{A}_1(0, \varphi, 0) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0)\big] \\ -\mathcal{A}_1\left(t, y_t, \int_0^t a(t, s, y_s)ds\right) \\ -\int_0^t \mathscr{A}\mathcal{T}_q(t - s)\mathcal{A}_1\left(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\right)ds \\ -\int_0^t \int_0^s \mathcal{T}_q(t - s)\mathcal{A}_2(t - s)\mathcal{A}_1\left(\tau, y_\tau, \int_0^\tau a(\tau, \mu, y_\mu)d\mu\right)d\tau ds \\ +\int_0^t \mathcal{T}_q(t - s)Bu(s)ds + \int_0^t \mathcal{T}_q(t - s)\left[\int_{-\infty}^s h(\tau, y_{\varrho(\tau, y_\tau)})dW(\tau)\right]ds \\ +\int_0^t \int_Z \mathcal{T}_q(t - s)g(s, y_s, z)\widetilde{\mathcal{N}}(ds, dz) + \sum_{0 < t_p < t} \mathcal{T}_q(t - t_p)\mathcal{I}_p(y_{t_p}) \\ +\sum_{0 < t_p < t} \mathcal{T}_q(t - t_p)\overline{\mathcal{I}}_p(y_{t_p}), \quad p = 1, 2, \cdots, n. \end{cases}$$

$$(3.1)$$

(iii) $y_0(t) = \varphi(t) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(t), \quad t \in (-\infty, 0], \quad y'(0) = \xi.$

In order to obtain the main result, we make the following hypotheses:

(H1) The operator families $\mathcal{R}_q(t)$ and $\mathcal{T}_q(t)$ are compact for all $t > 0$ and there exists a positive constant $\mathcal{M}$ such that

$$\sup_{t \in \mathscr{I}} \|\mathcal{R}_q(t)\| \vee \sup_{t \in \mathscr{I}} \|\mathcal{T}_q(t)\| \leq \mathcal{M}.$$

(H2) The functions $m_1 : \mathscr{B}^m \to \mathscr{B}$ is continuous, satisfy the Lipschitz conditions and we can find a positive constant $\mathcal{N}_{m_1}$ such that

$$\mathbb{E}\|m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})\|_{\mathcal{H}}^2 \leq \mathcal{N}_{m_1},$$

for all $(y_{t_1}, y_{t_2}, \cdots, y_{t_m}) \in \mathscr{B}^m$ and $t \in \mathscr{I}$.

(H3) The function $\mathcal{A}_1 : \mathscr{I} \times \mathscr{B} \times \mathcal{H} \to \mathcal{H}$ is continuous there exist positive constants $\mathcal{N}_{\mathcal{A}_1}, \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta}$ such that $\mathcal{A}_1$ is $\mathcal{H}_\beta$-valued and

$$\mathbb{E}\|\mathcal{A}_1(t, x, y)\|_{\mathcal{H}}^2 \leq \mathcal{N}_{\mathcal{A}_1}(1 + \|x\|_{\mathscr{B}}^2 + \|y\|_{\mathcal{H}}^2), \quad \text{for} \quad t \in \mathscr{I}, x \in \mathscr{B}, y \in \mathcal{H},$$

$$\mathbb{E}\|(-\mathscr{A})^\beta \mathcal{A}_1(t, x, y)\|_{\mathcal{H}}^2 \leq \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta}(1 + \|x\|_{\mathscr{B}}^2 + \|y\|_{\mathcal{H}}^2), \quad \text{for} \quad t \in \mathscr{I}, x \in \mathscr{B}, y \in \mathcal{H}.$$

(H4) For each $x \in \mathscr{B}, \overline{K}(t) = \lim_{a \to \infty} \int_{-a}^0 h(t, x)dW(s)$ exists and is continuous. And also we can find a positive constant $\mathcal{N}_k$ such that

$$\mathbb{E}\|\overline{K}(t)\|_{\mathcal{H}}^2 \leq \mathcal{N}_k.$$

(H5) The functions $h : \mathscr{I} \times \mathscr{B} \to \mathscr{L}(\mathcal{K}, \mathcal{H})$ and $g : \mathscr{I} \times \mathscr{B} \times Z \to \mathcal{H}$ are satisfies the following properties:
  (i) The function $h : \mathscr{I} \times \mathscr{B} \to \mathscr{L}(\mathcal{K}, \mathcal{H})$ is a continuous and measurable function.
  (ii) The non-linear function g is a Borel measurable function which satisfy the Lipschitz continuity condition, and we can find constants $\widetilde{\mathcal{N}}_g$ and L_g such that

$$\int_Z \mathbb{E}\|g(t, x, y)\|_{\mathcal{H}}^2 \kappa(dy) \leq \widetilde{\mathcal{N}}_g(1 + \|x\|_{\mathscr{B}}^2),$$

$$\int_Z \mathbb{E}\|g(t,x_1,y) - g(t,x_2,y)\|^2_{\mathcal{H}}\kappa(dy) \leq \widetilde{\mathcal{N}}_g\|x_1 - x_2\|^2_{\mathscr{B}},$$

$$\int_Z \mathbb{E}\|g(t,x,y)\|^4_{\mathcal{H}}\kappa(dy) \leq L_g(1 + \|x\|^4_{\mathscr{B}}),$$

$$\int_Z \mathbb{E}\|g(t,x_1,y) - g(t,x_2,y)\|^4_{\mathcal{H}}\kappa(dy) \leq L_g\|x_1 - x_2\|^4_{\mathscr{B}}, \quad \text{for all} \quad x, x_1, x_2 \in \mathscr{B}, t \in \mathscr{I}.$$

(iii) There are positive integrable functions $m, n \in \mathscr{L}^1(\mathscr{I})$ and continuous increasing functions $\mathcal{N}_h, \mathcal{N}_g : [0, \infty) \rightarrow (0, \infty)$ such that for all $(t, x) \in \mathscr{I} \times \mathscr{B}$, we have

$$\int_0^t \mathbb{E}\|h(t,x)\|^2_{\mathcal{H}}ds \leq m(t)\mathcal{N}_h\|x\|^2_{\mathscr{B}}, \quad \lim_{\tau \to \infty} \inf \frac{\mathcal{N}_h(\tau)}{\tau} = v_1 < \infty,$$

$$\int_0^t \mathbb{E}\|g(t,x,y)\|^2_{\mathcal{H}}ds \leq n(t)\mathcal{N}_g(1 + \|x\|^2_{\mathscr{B}}), \quad \lim_{\tau \to \infty} \inf \frac{\mathcal{N}_g(\tau)}{\tau} = v_2 < \infty.$$

(H6) The functions $\mathcal{I}_p$ and $\overline{\mathcal{I}}_p : \mathcal{H} \rightarrow \mathcal{H}$ are continuous and we can find positive constants $\mathcal{N}_{\mathcal{I}_p}, \mathcal{N}_{\overline{\mathcal{I}}_p}, p = 1, 2, \cdots, n$ such that

$$\mathbb{E}\|\mathcal{I}_p(x) - \mathcal{I}_p(y)\|^2_{\mathcal{H}} \leq \mathcal{N}_{\mathcal{I}_p}\|x - y\|^2_{\mathscr{B}}, \quad x, y \in \mathscr{B}, \quad p = 1, 2, \cdots, n,$$

$$\mathbb{E}\|\overline{\mathcal{I}}_p(x) - \overline{\mathcal{I}}_p(y)\|^2_{\mathcal{H}} \leq \mathcal{N}_{\overline{\mathcal{I}}_p}\|x - y\|^2_{\mathscr{B}}, \quad x, y \in \mathscr{B}, \quad p = 1, 2, \cdots, n.$$

(H7) The functions $\mathcal{I}_p$ and $\overline{\mathcal{I}}_p : \mathcal{H} \rightarrow \mathcal{H}$ are continuous and we can find positive increasing functions $\theta_p, \overline{\theta}_p : [0, \infty) \rightarrow (0, \infty), p = 1, 2, \cdots, n$ such that for all $y \in \mathscr{B}$ and $p = 1, 2, \cdots, n$, we have

$$\mathbb{E}\|\mathcal{I}_p(y)\|^2_{\mathcal{H}} \leq \theta_p\mathcal{N}_{\mathcal{I}_p}\|y\|^2_{\mathscr{B}}, \quad \lim_{\tau \to \infty} \inf \frac{\mathcal{N}_{\mathcal{I}_p}(\tau)}{\tau} = v_3 < \infty,$$

$$\mathbb{E}\|\overline{\mathcal{I}}_p(y)\|^2_{\mathcal{H}} \leq \overline{\theta}_p\mathcal{N}_{\overline{\mathcal{I}}_p}\|y\|^2_{\mathscr{B}}, \quad \lim_{\tau \to \infty} \inf \frac{\mathcal{N}_{\overline{\mathcal{I}}_p}(\tau)}{\tau} = v_4 < \infty.$$

(H8) The function $\mathcal{A}_1 : \mathscr{I} \times \mathscr{B} \times \mathcal{H} \rightarrow \mathcal{H}$ is continuous and there exists positive constants $\mathcal{N}_{\mathcal{A}_1,(-\mathscr{A})^\beta} > 0$ such that for all $(t, x_j) \in \mathscr{I} \times \mathscr{B}, j = 1, 2$

$$\|(-\mathscr{A})^\beta \mathcal{A}_1(t, x_1, y_1) - (-\mathscr{A})^\beta \mathcal{A}_1(t, x_2, y_2)\|^2_{\mathcal{H}}$$

$$\leq \mathcal{N}_{\mathcal{A}_1,(-\mathscr{A})^\beta}\|x_1 - x_2\|^2_{\mathscr{B}} + \mathcal{N}_{\mathcal{A}_1,(-\mathscr{A})^\beta}\|y_1 - y_2\|^2_{\mathcal{H}}, \quad x_1, x_2 \in \mathscr{B}, \, y_1, y_2 \in \mathcal{H}.$$

(H9) $a : \mathscr{I} \times \mathscr{I} \times \mathscr{B} \rightarrow \mathcal{H}$ is continuous and we can find constant $\mathcal{N}_a > 0$ and $\mathcal{N}_a^* > 0$ to ensure that

$$\left\|\int_0^t a(t, s, x)ds\right\|^2_{\mathcal{H}} \leq \mathcal{N}_a\|x\|^2_{\mathcal{H}},$$

$$\left\|\int_0^t \left[a(t, s, x_1) - a(t, s, x_2)\right]ds\right\|^2_{\mathcal{H}} \leq \mathcal{N}_a^*\|x_1 - x_2\|^2_{\mathcal{H}}.$$

In order to address the problem, it is convenient at the point to introduce two relevant operators and basic assumptions on these operators:

$$\Gamma_0^T = \int_0^T \mathcal{T}_q(T - s)BB^*\mathcal{T}_q^*(T - s)ds,$$

$$R(\lambda, \Gamma_0^T) = (\lambda I + \Gamma_0^T)^{-1}, \quad \text{for} \quad \lambda > 0.$$

where B^* denotes the adjoint of B and $\mathcal{T}_q^*(t)$ is the adjoint of $\mathcal{T}_q(t)$. It is straightforward that operator Γ_0^T is a linear bounded operator.

(H10) $\lambda R(\lambda, \Gamma_0^T) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

Also, we note that the assumption $(H10)$ is equivalent to the fact that the linear fractional control system

$$D_t^q y(t) = \mathscr{A} y(t) + Bu(t),$$
$$y(0) = y_0 \in \mathcal{H}, \quad y'(0) = 0,$$

is approximately controllable on $\mathscr{I}$.

Definition 3.2. [19] *The model (1.1)-(1.5) is said to be approximately controllable on the interval $\mathscr{I}$ if $\overline{\mathfrak{R}(T, \varphi, \xi)} = \mathcal{H}$, where $\overline{\mathfrak{R}(T, \varphi, \xi)}$ is the closure of the reachable set.*

Lemma 6. [27] *For any $\overline{y} \in \mathscr{L}_2(\mathfrak{F}_T, \mathcal{H})$, there exists $\overline{\varphi} \in \mathscr{L}_{\mathfrak{F}}^2(\Omega, \mathscr{L}_2(0, T, \mathscr{L}_2^0))$ such that $\overline{y}_T = \mathbb{E}\overline{y}_T + \int_0^T \overline{\varphi}(s)dW(s)$.*
For any $\lambda > 0, p = 1, 2, \cdots, n$ and $\overline{y} \in \mathscr{L}_2(\mathfrak{F}_T, \mathcal{H})$, we define the control function

$$
\begin{aligned}
u^\lambda(t) = {} & B^* \mathcal{T}_q^*(T - t)(\lambda I + \Gamma_0^T)^{-1}\Bigg[\mathbb{E}\overline{y}_T + \int_0^T \overline{\varphi}(s)dW(s) \\
& - \mathcal{R}_q(T)\big[\varphi(0) + \mathcal{A}_1(0, \varphi, 0) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0)\big] \\
& + \mathcal{A}_1\Big(t, y_t, \int_0^t a(t, s, y_s)ds\Big)\Bigg] \\
& + B^* \mathcal{T}_q^*(T - t) \int_0^t (\lambda I + \Gamma_0^T)^{-1} \mathscr{A} \mathcal{T}_q(T - s)\mathcal{A}_1\Big(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\Big)ds \\
& + B^* \mathcal{T}_q^*(T - t) \int_0^t \int_0^s (\lambda I + \Gamma_0^T)^{-1} \mathcal{T}_q(T - s)\mathcal{A}_2(t - s) \\
& (\times)\mathcal{A}_1\Big(\tau, y_\tau, \int_0^\tau a(\tau, \mu, y_\mu)d\mu\Big)d\tau ds \\
& - B^* \mathcal{T}_q^*(T - t) \int_0^t (\lambda I + \Gamma_0^T)^{-1} \mathcal{T}_q(T - s)Bu^\lambda(s)ds \\
& - B^* \mathcal{T}_q^*(T - t) \int_0^t (\lambda I + \Gamma_0^T)^{-1} \mathcal{T}_q(T - s)\bigg[\int_{-\infty}^s h(\tau, y_{\varrho(\tau, y_\tau)})dW(\tau)\bigg]ds \\
& - B^* \mathcal{T}_q^*(T - t) \int_0^t \int_Z (\lambda I + \Gamma_0^T)^{-1} \mathcal{T}_q(T - s)g(s, y_s, z)\widetilde{N}(ds, dz) \\
& - B^* \mathcal{T}_q^*(T - t)(\lambda I + \Gamma_0^T)^{-1} \sum_{0 < t_p < t} \mathcal{T}_q(T - t_p)\mathcal{I}_p(y_{t_p}) \\
& - B^* \mathcal{T}_q^*(T - t)(\lambda I + \Gamma_0^T)^{-1} \sum_{0 < t_p < t} \mathcal{T}_q(T - t_p)\overline{\mathcal{I}}_p(y_{t_p}).
\end{aligned}
\tag{3.3}
$$

Theorem 3.1. *If the assumptions $(H1) - (H9)$ are satisfied. Further, suppose that for all $\lambda > 0$ and $p = 1, 2, \cdots, n$, then the control system (1.1)-(1.5) has mild*

solution on $\mathscr{I}$ provided that

$$9\left(1 + \frac{11\mathcal{M}^4\mathcal{M}_B^4 T^2}{\lambda^2}\right)\left[4(1 + \mathcal{N}_a)\left(\mathcal{N}_{\mathcal{A}_1} + \mathcal{M}^2\mathcal{N}_{\mathcal{A}_1,(-\mathscr{A})^\beta}\frac{T^{2q\beta}}{q^2\beta^2}\right)\left(1 + \int_0^T \mu(s)ds\right)K_1^{*2}\right.$$

$$+ 8\mathcal{M}^2 T^2 K_1^{*2}\left[Tr(Q)\mathcal{N}_h \sup_{s\in\mathscr{I}} m(s) + (\widetilde{\mathcal{N}}_g + \sqrt{L_g})\sup_{s\in\mathscr{I}} n(s)\right]$$

$$\left.+ 4\mathcal{M}^2 K_1^{*2}\left[\sum_{p=1}^n \mathcal{N}_{\mathcal{I}_p}\sup_{s\in\mathscr{I}}\theta_p(s) + \sum_{p=1}^n \mathcal{N}_{\overline{\mathcal{I}}_p}\sup_{s\in\mathscr{I}}\overline{\theta}_p(s)\right]\right] < 1, \tag{A}$$

Proof. Consider the space $C((-\infty, T], \mathcal{H})$ are all continuous with $\mathcal{H}$- valued stochastic process $\{\zeta(t) : t \in (-\infty, T]\}$ and $\mathscr{B}_T = \{y : y \in C((-\infty, T], \mathcal{H}), y_0 \in \mathscr{B}\}$. Let $\|\cdot\|_T$ be the semi-norm defined by $\|y\|_T \leq \|y_0\|_{\mathcal{B}_h} + \sup_{0\leq s\leq T}(\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}}, y \in \mathscr{B}_T$. For any $\lambda > 0$ and $p = 1, 2, \cdots, n$, defined the operator $\Phi^\lambda : \mathscr{B}_T \to \mathscr{B}_T$ by

$$(\Phi^\lambda y)(t) = \begin{cases}
\varphi(t), t \in (-\infty, 0] \\
\mathcal{R}_q(t)\left[\varphi(0) + \mathcal{A}_1(0, \varphi, 0) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0)\right] \\
-\mathcal{A}_1\left(t, y_t, \int_0^t a(t, s, y_s)ds\right) \\
-\int_0^t \mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\left(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\right)ds \\
-\int_0^t \int_0^s \mathcal{T}_q(t-s)\mathcal{A}_2(t-s)\mathcal{A}_1\left(\tau, y_\tau, \int_0^\tau a(\tau, \mu, y_\mu)d\mu\right)d\tau ds \\
+\int_0^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds + \int_0^t \mathcal{T}_q(t-s)\int_{-\infty}^s h(\tau, y_{\varrho(\tau, y_\tau)})dW(\tau)ds \\
+\int_0^t \int_Z \mathcal{T}_q(t-s)g(s, y_s, z)\widetilde{N}(ds, dz) \\
+\sum_{0<t_p<t} \mathcal{T}_q(t-t_p)\mathcal{I}_p(y_{t_p}) + \sum_{0<t_p<t} \mathcal{T}_q(t-t_p)\overline{\mathcal{I}}_p(y_{t_p}), \quad t \in \mathscr{I}.
\end{cases}$$

$\square$

From the hypothesis $(H3)$ and using Holder's inequality, we get

$$\mathbb{E}\left\|\int_0^t \mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\left(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\right)ds\right\|_{\mathcal{H}}^2$$

$$\leq \mathcal{M}^2\left(\int_0^t (t-s)^{q\beta-1}ds\right)\left(\int_0^t (t-s)^{q\beta-1}\right.$$

$$(\times)\mathbb{E}\left\|(-\mathscr{A})^\beta\mathcal{A}_1\left(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\right)\right\|_{\mathcal{H}}^2 ds\Big)$$

$$\leq \mathcal{M}^2\mathcal{N}_{\mathcal{A}_1,(-\mathscr{A})^\beta}\frac{T^{q\beta}}{q\beta}\int_0^t (t-s)^{q\beta-1}\left(1 + \|y_s\|_{\mathscr{B}}^2 + \left\|\int_0^s a(s, \tau, y_\tau)d\tau\right\|_{\mathscr{B}}^2\right)ds.$$

By Bochner's theorem and Lemma 2.1, $\mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\left(s, y_s, \int_0^s a(s, \tau, y_\tau)d\tau\right)$ is integrable on $\mathscr{I}$. Next, we prove that Φ^λ has a fixed point, by Sadovskii's theorem,

which is then a mild solution of the given system (1.1)-(1.5) . Define

$$
v(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0] \\ \mathcal{R}_q(t)\big[\varphi(0) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0)\big], & t \in \mathscr{I}. \end{cases}
$$

It is clear that $x \in \mathscr{B}$. Let $y(t) = x_t + v_t, t \in (-\infty, T]$. Then y satisfies (1.1)-(1.5) if and only if $v_0 = \varphi$ and

$$
x(t) = \begin{cases} \mathcal{R}_q(t)\mathcal{A}_1(0, \varphi, 0) - \mathcal{A}_1\left(t, x_t + v_t, \int_0^t a(t, s, x_s + v_s)ds\right) \\ - \int_0^t \mathscr{A}\mathcal{T}_q(t - s)\mathcal{A}_1\left(s, x_s + v_s, \int_0^s a(s, \tau, x_\tau + v_\tau)d\tau\right)ds \\ - \int_0^t \int_0^s \mathcal{T}_q(t - s)\mathcal{A}_2(t - s)\mathcal{A}_1\left(\tau, x_\tau + v_\tau, \int_0^\tau a(\tau, \xi, x_\xi + v_\xi)d\xi\right)d\tau ds \\ + \int_0^t \mathcal{T}_q(t - s)Bu^\lambda(s)ds + \int_0^t \mathcal{T}_q(t - s) \\ (\times)\left[\int_{-\infty}^s g(\tau, x_{\varrho(\tau, x_\tau + v_\tau)} + v_{\varrho(\tau, x_\tau + v_\tau)})dW(\tau)\right]ds \\ + \int_0^t \int_z \mathcal{T}_q(t - s)g(s, x_s + v_s, z)\widetilde{N}(ds, dz) + \sum_{0 < t_p < t} \mathcal{T}_q(t - t_p)\mathcal{I}_p(x_{t_p} + v_{t_p}) \\ + \sum_{0 < t_p < t} \mathcal{T}_q(t - t_p)\overline{\mathcal{I}}_p(x_{t_p} + v_{t_p}), \quad p = 1, 2, \cdots, n. \end{cases}
$$

Where $u^\lambda(t)$ is defined (3.3). Let $\mathscr{B}_T^0 = \{x \in \mathscr{B}_T : x_0 = 0 \in \mathscr{B}\}$. For any $x \in \mathscr{B}_T^0$, can be defined by

$$
\|x\|_{\mathscr{B}_T^0} = \sup_{s \in \mathscr{I}} \left(\mathbb{E}\|x\|^2\right)^{\frac{1}{2}} + \|x_0\|_{\mathscr{B}} = \sup_{s \in \mathscr{I}} \left(\mathbb{E}\|x\|^2\right)^{\frac{1}{2}}, \quad x \in \mathscr{B}_T^0.
$$

As a result, $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. Set $B_r = \{x \in \mathscr{B}_T^0 : \|x\|^2 \leq r\}$ for each positive number r. Then for each r, B_r is a bounded closed convex set in $\mathscr{B}_T^0$. For $x \in B_r$, then by Lemma 2.1, we have

$$
\begin{aligned}
&\mathbb{E}\|x_t + v_t\|_{\mathscr{B}}^2 \\
&\leq 2\big[\mathbb{E}\|x_t\|_{\mathscr{B}}^2 + \mathbb{E}\|v_t\|_{\mathscr{B}}^2\big] \\
&\leq 4\left[K_2^{*2}\mathbb{E}\|x_0\|_{\mathscr{B}}^2 + K_1^{*2} \sup_{0 \leq s \leq t} \mathbb{E}\|x(s)\|^2 + K_2^{*2}\mathbb{E}\|v_0\|_{\mathscr{B}}^2 + K_1^{*2} \sup_{0 \leq s \leq t} \mathbb{E}\|v(s)\|^2\right] \\
&\leq 4K_1^{*2}r + C_1 = \widetilde{r}, \tag{3.4}
\end{aligned}
$$

where $C_1 = 4\big(K_1^{*2}\mathcal{M}^2[\mathbb{E}\|\varphi(0)\|_{\mathscr{B}}^2 + \mathcal{N}_{m_1}] + K_2^{*2}\mathbb{E}\|\varphi\|_{\mathscr{B}}^2\big)$.

$$\mathbb{E}\|x_{\varrho(t,x_t+v_t)} + v_{\varrho(t,x_t+v_t)}\|^2$$
$$\leq 2(\mathbb{E}\|x_{\varrho(t,x_t+v_t)}\|^2 + \mathbb{E}\|v_{\varrho(t,x_t+v_t)}\|^2)$$
$$\leq 4\Bigg(K_1^{*2} \sup_{\substack{0\leq s\leq \max(0,t)\\ t\in\mathscr{R}(\varrho^-)\cup\mathscr{I}}} \mathbb{E}\|x(s)\|_{\mathcal{H}}^2 + (K_2^* + J^\varphi)^2\mathbb{E}\|x_0\|_{\mathscr{B}}^2\Bigg)$$
$$\quad + 4\Bigg(K_1^{*2} \sup_{\substack{0\leq s\leq \max(0,t)\\ t\in\mathscr{R}(\varrho^-)\cup\mathscr{I}}} \mathbb{E}\|v(s)\|_{\mathcal{H}}^2 + (K_2^* + J^\varphi)^2\mathbb{E}\|v_0\|_{\mathscr{B}}^2\Bigg)$$
$$\leq 4\Bigg(K_1^{*2}r + K_1^{*2}\mathbb{E}\|\mathcal{R}_q(t)(\varphi(0) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0))\|^2 + (K_2^* + J^\varphi)^2\mathbb{E}\|\varphi\|_{\mathscr{B}}^2\Bigg)$$
$$\leq 4\Bigg(K_1^{*2}r + K_1^{*2}\mathcal{M}^2(\mathbb{E}\|\varphi(0)\|^2 + \mathcal{N}_m) + (K_2^* + J^\varphi)^2\mathbb{E}\|\varphi\|_{\mathscr{B}}^2\Bigg)$$
$$\leq 4K_1^{*2}r + C_2 = r^*, \tag{3.5}$$

where $C_2 = 4\big[K_1^{*2}\mathcal{M}^2(\mathbb{E}\|\varphi(0)\|^2 + \mathcal{N}_{m_1}) + (K_2^* + J^\varphi)^2\mathbb{E}\|\varphi\|_{\mathscr{B}}^2\big]$.

Define the map Φ from $\mathscr{B}_T^0$ into itself. Therefore, we have

$$(\Phi x)(t) = \begin{cases} 0, \quad t\in(-\infty, 0], \\ \mathcal{R}_q(t)\mathcal{A}_1(0, \varphi, 0) - \mathcal{A}_1\Big(t, x_t + v_t, \displaystyle\int_0^t a(t, s, x_s + v_s)ds\Big) \\ -\displaystyle\int_0^t \mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\Big(s, x_s + v_s, \int_0^s a(s, \tau, x_\tau + v_\tau)d\tau\Big)ds \\ -\displaystyle\int_0^t\int_0^s \mathcal{T}_q(t-s)\mathcal{A}_2(t-s)\mathcal{A}_1\Big(\tau, x_\tau + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu + v_\mu)d\mu\Big)d\tau ds \\ +\displaystyle\int_0^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds + \int_0^t \mathcal{T}_q(t-s) \\ \quad (\times)\Big[\displaystyle\int_{-\infty}^s h(s, x_{\varrho(\tau,x_\tau+v_\tau)} + v_{\varrho(\tau,x_\tau+v_\tau)})dW(\tau)\Big]ds \\ +\displaystyle\int_0^t\int_Z \mathcal{T}_q(t-s)g(s, x_s + v_s, z)\widetilde{N}(ds, dz) + \sum_{0<t_p<t} \mathcal{T}_q(t-t_p)\mathcal{I}_p(x_{t_p} + v_{t_p}) \\ \displaystyle\sum_{0<t_p<t} \mathcal{T}_q(t-t_p)\overline{\mathcal{I}}_p(x_{t_p} + v_{t_p}), \quad p = 1, 2, \cdots, n. \end{cases}$$

Then Φ is well defined on B_r for each $r > 0$. We see that the operator Φ^λ has a fixed point if and only if Φ has a fixed point. Thus, let us demonstrate that Φ^λ has a fixed point. Now, we split Φ as $\Phi_1 + \Phi_2$ where

$$(\Phi_1 x)(t) = \begin{cases} \mathcal{R}_q(t)\mathcal{A}_1(0, \varphi, 0) - \mathcal{A}_1\Big(t, x_t + v_t, \displaystyle\int_0^t a(t, s, x_s + v_s)ds\Big) \\ -\displaystyle\int_0^t \mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\Big(s, x_s + v_s, \int_0^s a(s, \tau, x_\tau + v_\tau)d\tau\Big)ds \\ +\displaystyle\sum_{0<t_p<t} \mathcal{T}_q(t-t_p)\mathcal{I}_p(x_{t_p} + v_{t_p}) + \sum_{0<t_p<t} \mathcal{T}_q(t-t_p)\overline{\mathcal{I}}_p(x_{t_p} + v_{t_p}), \\ \quad p = 1, 2, \cdots, n. \end{cases}$$

$$(\Phi_2 x)(t) = \begin{cases} -\int_0^t \int_0^s \mathcal{T}_q(t-s)\mathcal{A}_2(t-s)\mathcal{A}_1\left(\tau, x_\tau + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu + v_\mu)d\mu\right)d\tau ds \\ +\int_0^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds + \int_0^t \mathcal{T}_q(t-s) \\ (\times)\left[\int_{-\infty}^s h(\tau, x_{\varrho(\tau, x_\tau + v_\tau)} + v_{\varrho(\tau, x_\tau + v_\tau)})dW(\tau)\right]ds \\ +\int_0^t \int_Z \mathcal{T}_q(t-s)g(s, x_s + v_s, z)\widetilde{N}(ds, dz). \end{cases}$$

The proof of the theorem is technical. Therefore, we need the following theorem.

Theorem 3.2. *If the above hypothesis (H1) - (H9) are satisfied. Then Φ_1 is a contraction mapping and Φ_2 is compact.*

Proof. To prove this theorem, we first prove the following crucial lemmas. □

Lemma 7. *For each $\lambda > 0$ and suppose that $(H1) - (H9)$ hold. Then, we can find a positive number r such that $\Phi(B_r) \subset B_r$.*

Proof. Suppose that $\Phi(B_r) \not\subset B_r$. Then for each non-negative number r, there exists a function $x^r(\cdot) \in B_r$, but $\Phi(x^r) \notin B_r$, that is $\mathbb{E}\|(\Phi x^r)(t)\|_{\mathcal{H}}^2 > r$ for some $t = t(r) \in \mathscr{I}$. Then for each $\lambda > 0$ and $p = 1, 2, \cdots, n$, we have

$$\begin{aligned} r \leq& \mathbb{E}\|\Phi x^r\|_{\mathcal{H}}^2 \\ \leq& 9\mathbb{E}\|\mathcal{R}_q(t)\mathcal{A}_1(0, \varphi, 0)\|_{\mathcal{H}}^2 + 9\mathbb{E}\left\|\mathcal{A}_1\left(t, x_t^r + v_t, \int_0^t a(t, s, x_s^r + v_s)ds\right)\right\|_{\mathcal{H}}^2 \\ &+ 9\mathbb{E}\left\|\int_0^t \mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\left(s, x_s^r + v_s, \int_0^s a(s, \tau, x_\tau^r + v_\tau)d\tau\right)ds\right\|_{\mathcal{H}}^2 \\ &+ 9\mathbb{E}\left\|\int_0^t \int_0^s \mathcal{T}_q(t-s)\mathcal{A}_2(t-s)\mathcal{A}_1\left(\tau, x_\tau^r + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu^r + v_\mu)d\mu\right)d\tau ds\right\|_{\mathcal{H}}^2 \\ &+ 9\mathbb{E}\left\|\int_0^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds\right\|_{\mathcal{H}}^2 \\ &+ 9\mathbb{E}\left\|\int_0^t \mathcal{T}_q(t-s)\left[\int_{-\infty}^s h(\tau, x_{\varrho(\tau, x_\tau^r + v_\tau)}^r + v_{\varrho(\tau, x_\tau^r + v_\tau)})dW(\tau)\right]ds\right\|_{\mathcal{H}}^2 \\ &+ 9\mathbb{E}\left\|\int_0^t \int_Z \mathcal{T}_q(t-s)g(s, x_s^r + v_s, z)\widetilde{N}(ds, dz)\right\|_{\mathcal{H}}^2 \\ &+ 9\sum_{0 < t_p < t} \mathbb{E}\left\|\mathcal{T}_q(t-t_p)\mathcal{I}_p(x_{t_p}^r + v_{t_p})\right\|^2 + 9\sum_{0 < t_p < t} \mathbb{E}\left\|\mathcal{T}_q(t-t_p)\overline{\mathcal{I}}_p(x_{t_p}^r + v_{t_p})\right\| \\ =& 9\sum_{i=1}^9 J_i. \end{aligned} \tag{3.6}$$

From Lemma (2.1), $(H1) - (H9)$, (3.4), (3.5) and Holder's inequality, we get:

$$J_1 \leq \mathcal{M}^2 \mathbb{E}\|\mathcal{A}_1(0, \varphi, 0)\|_{\mathcal{H}}^2$$
$$\leq \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1}(1 + \|\varphi\|_{\mathscr{B}}^2).$$
$$J_2 \leq \mathbb{E}\left\| \mathcal{A}_1\left(t, x_t^r + v_t, \int_0^t a(t, s, x_s^r + v_s)ds \right) \right\|_{\mathcal{H}}^2$$
$$\leq \mathcal{N}_{\mathcal{A}_1}\left(1 + \|x_t^r + v_t\|_{\mathscr{B}}^2 + \left\| \int_0^t a(t, s, x_s^r + v_s)ds \right\|_{\mathcal{H}}^2 \right)$$
$$\leq \mathcal{N}_{\mathcal{A}_1}\left(1 + 4K_1^{*2}r + C_1 + \mathcal{N}_a(4K_1^{*2}r + C_1) \right)$$
$$\leq 4\mathcal{N}_{\mathcal{A}_1}[K_1^{*2}r(1 + \mathcal{N}_a) + C_3], \quad \text{where} \quad C_3 = \mathcal{N}_{\mathcal{A}_1}(1 + C_1(1 + \mathcal{N}_a)).$$
$$J_3 \leq \mathbb{E}\left\| \int_0^t \mathscr{A}\mathcal{T}_q(t-s)\mathcal{A}_1\left(s, x_s^r + v_s, \int_0^s a(s, \tau, x_\tau^r + v_\tau)d\tau \right)ds \right\|_{\mathcal{H}}^2$$
$$\leq \mathcal{M}^2 \frac{T^{q\beta}}{q\beta} \int_0^t (t-s)^{q\beta-1}\mathbb{E}\left\| (-\mathscr{A})^\beta \mathcal{A}_1\left(s, x_s^r + v_s, \int_0^s a(s, \tau, x_\tau^r + v_\tau)d\tau \right) \right\|_{\mathcal{H}}^2 ds$$
$$\leq \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{(q\beta)^2}\left(4K_1^{*2}r(1 + \mathcal{N}_a) + C_4 \right), \quad \text{where} \quad C_4 = 1 + C_1(1 + \mathcal{N}_a).$$
$$J_4 \leq \mathbb{E}\left\| \int_0^t \int_0^s \mathcal{T}_q(t-s)\mathcal{A}_2(t-s)\mathcal{A}_1\left(\tau, x_\tau^r + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu^r + v_\mu)d\mu \right)d\tau ds \right\|_{\mathcal{H}}^2$$
$$\leq \mathcal{M}^2 \int_0^t \int_0^s (t-s)^{2(q\beta-1)}\mu(t-s)$$
$$(\times)\mathbb{E}\left\| (-\mathscr{A})^\beta \mathcal{A}_1\left(\tau, x_\tau^r + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu^r + v_\mu)d\mu \right) \right\|^2 d\tau ds$$
$$\leq \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2\beta^2}\left(4K_1^{*2}r(1 + \mathcal{N}_a) + C_4 \right) \int_0^T \mu(s)ds.$$
$$J_5 \leq \mathbb{E}\left\| \int_0^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds \right\|_{\mathcal{H}}^2$$
$$\leq \mathcal{M}^2 \mathcal{M}_B^2 T \int_0^t \mathbb{E}\|u^\lambda(s)\|^2 ds, \quad \text{where}\|B\| = \mathcal{M}_B$$
$$\leq \mathcal{M}^2 \mathcal{M}_B^2 T^2 \mathbb{E}\|u^\lambda(s)\|,$$

where

$$\mathbb{E}\|u^\lambda(s)\|^2 \leq 11\frac{\mathcal{M}_B^2 \mathcal{M}^2}{\lambda^2}\left[2\mathbb{E}\|\overline{y}_T\|^2 + 2\int_0^T \mathbb{E}\|\overline{\varphi}(s)\|^2 ds + \mathcal{M}^2(\|\varphi\|_B^2 + \overline{B}) \right.$$
$$+ \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1}(1 + \|\varphi\|_{\mathscr{B}}^2) + \mathcal{N}_{\mathcal{A}_1}\left(4K_1^{*2}r(1 + \mathcal{N}_a) + C_3 \right)$$
$$+ \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2\beta^2}\left(1 + \int_0^T \mu(s)ds \right)\left[4K_1^{*2}r(1 + \mathcal{N}_a) + C_4 \right]$$
$$\left. + \mathcal{M}^2 T^2\left(2\mathcal{N}_k + 2Tr(Q)\mathcal{N}_h(4K_1^{*2}r + C_2) \sup_{t \in \mathscr{I}} m(s) \right) \right)$$

$$+ \mathcal{M}^2 T^2 (2\widetilde{\mathcal{N}}_g + 2\sqrt{L_g})(1 + 4K_1^{*2}r + C_1) \sup_{s \in \mathscr{I}} n(s)$$
$$+ \mathcal{M}^2 \sum_{p=1}^{n} (4K_1^{*2}r + C_1) \Big(\mathcal{N}_{\mathcal{I}_p} \sup_{s \in \mathscr{I}} \theta_p(s) + \mathcal{N}_{\overline{\mathcal{I}}_p} \sup_{s \in \mathscr{I}} \overline{\theta}_p(s) \Big) \bigg],$$

where

$$\overline{B} = \|\varphi\|_{\mathscr{B}}^2 + \|m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(t)\|_{\mathcal{H}}^2 + \|\xi\|_{\mathscr{B}}^2.$$

Thus, we have

$$J_5 \leq (\mathcal{M}^2 \mathcal{M}_B^2 T^2) \Big(\frac{11}{\lambda^2} \mathcal{M}^2 \mathcal{M}_B^2 \Big) \mathcal{N}_R,$$

and

$$\mathcal{N}_R = 2\mathbb{E}\|\overline{y}_T\|^2 + 2\int_0^T \mathbb{E}\|\overline{\varphi}(s)\|^2 ds + \mathcal{M}^2 (\|\varphi\|_B^2 + \overline{B}) + \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1}(1 + \|\varphi\|_{\mathscr{B}}^2)$$
$$+ \mathcal{N}_{\mathcal{A}_1} \big(4K_1^{*2}r(1 + \mathcal{N}_a) + C_3 \big) + \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2 \beta^2} \Big(1 + \int_0^T \mu(s) ds \Big)$$
$$(\times) \big[4K_1^{*2}r(1 + \mathcal{N}_a) + C_4 \big] + \mathcal{M}^2 T^2 \Big(2\mathcal{N}_k + 2Tr(Q)\mathcal{N}_h (4K_1^{*2}r + C_2) \sup_{t \in \mathscr{I}} m(s) \Big)$$
$$+ \mathcal{M}^2 T^2 (2\mathcal{N}_g + 2\sqrt{L_g}) \widetilde{\mathcal{N}}_g (1 + 4K_1^{*2}r + C_1) \sup_{s \in \mathscr{I}} n(s)$$
$$+ \mathcal{M}^2 \sum_{p=1}^{n} (4K_1^{*2}r + C_1) \Big(\mathcal{N}_{\mathcal{I}_p} \sup_{s \in \mathscr{I}} \theta_p(s) + \mathcal{N}_{\overline{\mathcal{I}}_p} \sup_{s \in \mathscr{I}} \overline{\theta}_p(s) \Big).$$

$$J_6 \leq \mathbb{E} \bigg\| \int_0^t \mathcal{T}_q(t-s) \bigg[\int_{-\infty}^s h(\tau, x_{\varrho(\tau, x_\tau^r + v_\tau)}^r + v_{\varrho(\tau, x_\tau^r + v_\tau)}) dW(\tau) \bigg] ds \bigg\|_{\mathcal{H}}^2$$
$$\leq \mathcal{M}^2 T^2 \bigg[2\mathcal{N}_k + 2Tr(Q) \int_0^t \mathbb{E}\|h(\tau, x_{\varrho(\tau, x_\tau^r + v_\tau)}^r + v_{\varrho(\tau, x_\tau^r + v_\tau)})\|_{\mathscr{B}}^2 d\tau \bigg]$$
$$\leq \mathcal{M}^2 T^2 \bigg[2\mathcal{N}_k + 2Tr(Q)\mathcal{N}_h (4K_1^{*2}r + C_2) \sup_{s \in \mathscr{I}} m(s) \bigg].$$

$$J_7 \leq \mathbb{E} \bigg\| \int_0^t \int_Z \mathcal{T}_q(t-s) g(s, x_s^r + v_s, z) \widetilde{N}(ds, dz) \bigg\|^2$$
$$\leq 2\mathcal{M}^2 \int_0^t \int_Z \mathbb{E}\|g(s, x_s^r + v_s, z)\|_{\mathcal{H}}^2 \kappa(dz) ds$$
$$+ 2\mathcal{M}^2 \bigg(\int_0^t \int_Z \mathbb{E}\|g(s, x_s^r + v_s, z)\|_{\mathcal{H}}^4 \kappa(dz) ds \bigg)^{\frac{1}{2}}$$
$$\leq \mathcal{M}^2 (2\widetilde{\mathcal{N}}_g + 2\sqrt{L_g}) \int_0^t (1 + \|x_s^r + v_s\|_{\mathscr{B}}^2) ds$$
$$\leq \mathcal{M}^2 T^2 (2\widetilde{\mathcal{N}}_g + 2\sqrt{L_g})(1 + 4K_1^{*2}r + C_1) \sup_{s \in \mathscr{I}} n(s).$$

$$J_8 \leq \mathbb{E} \left\| \sum_{0 < t_p < t} \mathcal{T}_q(t - t_p) \mathcal{I}_p(x_{t_p}^r + v_{t_p}) \right\|^2$$

$$\leq \mathcal{M}^2 \sum_{p=1}^{n} \theta_p(s) \mathcal{N}_{\mathcal{I}_p} \left(\|x_{t_p} + v_{t_p}^r\|^2 \right)$$

$$\leq \mathcal{M}^2 \sum_{p=1}^{n} \mathcal{N}_{\mathcal{I}_p} (4K_1^{*2} r + C_1) \sup_{s \in \mathscr{I}} \theta_p(s).$$

$$J_9 \leq \mathcal{M}^2 \sum_{p=1}^{n} \mathcal{N}_{\overline{\mathcal{I}}_p} (4K_1^{*2} r + C_1) \sup_{s \in \mathscr{I}} \overline{\theta}_p(s).$$

Combining the estimations $(J_1) - (J_9)$ together with (3.6) we have

$$r \leq \widetilde{\mathcal{N}}_s + 36(1 + \mathcal{N}_a) \left(\mathcal{N}_{\mathcal{A}_1} + \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2 \beta^2} \left(1 + \int_0^T \mu(s) ds \right) \right) K_1^{*2} r$$

$$+ 72 \mathcal{M}^2 T^2 Tr(Q) \mathcal{N}_h (K_1^{*2} r) \sup_{s \in \mathscr{I}} m(s) + 72 \mathcal{M}^2 T^2 (\widetilde{\mathcal{N}}_g + \sqrt{L_g})(K_1^{*2} r) \sup_{s \in \mathscr{I}} n(s)$$

$$+ 36 \mathcal{M}^2 \sum_{p=1}^{n} \mathcal{N}_{\mathcal{I}_p} (K_1^{*2} r) \sup_{s \in \mathscr{I}} \theta_p(s) + \quad 36 \mathcal{M}^2 \sum_{p=1}^{n} \mathcal{N}_{\overline{\mathcal{I}}_p} (K_1^{*2} r) \sup_{s \in \mathscr{I}} \overline{\theta}_p(s)$$

$$+ \frac{99}{\lambda^2} (\mathcal{M}^4 \mathcal{M}_B^4 T^2) \mathcal{N}_R,$$

where

$$\widetilde{\mathcal{N}}_s = 9 \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1} (1 + \|\varphi\|_{\mathscr{B}}^2) + 9 \mathcal{N}_{\mathcal{A}_1} C_3 + 9 \mathcal{M}^2 C_4 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2 \beta^2}$$

$$+ 9 \mathcal{M}^2 C_1 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2 \beta^2} \int_0^T \mu(s) ds + 18 \mathcal{M}^2 T^2 (\mathcal{N}_k + Tr(Q) \mathcal{N}_h C_2) \sup_{s \in \mathscr{I}} m(s)$$

$$+ 18 \mathcal{M}^2 T^2 (\widetilde{\mathcal{N}}_g + \sqrt{L_g})(1 + C_1) \sup_{s \in \mathscr{I}} n(s) + 9 \mathcal{M}^2 C_1 \sum_{p=1}^{n} \mathcal{N}_{\mathcal{I}_p} \sup_{s \in \mathscr{I}} \theta_p(s)$$

$$+ 9 \mathcal{M}^2 C_1 \sum_{p=1}^{n} \mathcal{N}_{\overline{\mathcal{I}}_p} \sup_{s \in \mathscr{I}} \overline{\theta}_p(s),$$

and C_1, C_2, C_3 and C_4 are independent of r defined priviously. Then, dividing both sides of above equation by r, and letting as $r \to \infty$, for $p = 1, 2, \cdots, n$

$$9 \left(1 + \frac{11 \mathcal{M}^4 \mathcal{M}_B^4 T^2}{\lambda^2} \right) \left[4(1 + \mathcal{N}_a) \left(\mathcal{N}_{\mathcal{A}_1} + \mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2 \beta^2} \right) \left(1 + \int_0^T \mu(s) ds \right) K_1^{*2}$$

$$+ 8 \mathcal{M}^2 T^2 K_1^{*2} \left[Tr(Q) \mathcal{N}_h \sup_{s \in \mathscr{I}} m(s) + (\widetilde{\mathcal{N}}_g + \sqrt{L_g}) \sup_{s \in \mathscr{I}} n(s) \right]$$

$$+ 4 \mathcal{M}^2 K_1^{*2} \left[\sum_{p=1}^{n} \mathcal{N}_{\mathcal{I}_p} \sup_{s \in \mathscr{I}} \theta_p(s) + \sum_{p=1}^{n} \mathcal{N}_{\overline{\mathcal{I}}_p} \sup_{s \in \mathscr{I}} \overline{\theta}_p(s) \right] \right] > 1,$$

which is a contradiction to our assumption (A). Thus for each $\lambda > 0$, there exists a non-negative number r such that $\Phi(B_r) \subset B_r$. $\qquad\square$

Lemma 8. *Let us assume* $(H1) - (H9)$ *holds. Then* Φ_1 *is contraction.*

Proof. Let $x, \overline{x} \in B_r$. Then for $p = 1, 2, \cdots, n$, by using $(H6), (H8)$ and $(H9)$ we have

$$
\mathbb{E}\|(\Phi_1 x)(t) - (\Phi_1 \overline{x})(t)\|_{\mathcal{H}}^2
$$

$$
\leq 4\mathcal{N}_{\mathcal{A}_1}(1 + \mathcal{N}_a^*) \sup_{0 \leq s \leq t} \mathbb{E}\|x(s) - \overline{x}(s)\|_{\mathcal{H}}^2 + 4\mathcal{M}^2 \mathcal{N}_{\mathcal{I}_p} \sup_{0 \leq s \leq t} \mathbb{E}\|x(s) - \overline{x}(s)\|_{\mathcal{H}}^2
$$

$$
+ 4\mathcal{M}^2 \mathcal{N}_{\overline{\mathcal{I}}_p} \sup_{0 \leq s \leq t} \mathbb{E}\|x(s) - \overline{x}(s)\|_{\mathcal{H}}^2 + 4\mathcal{M}^2 \mathcal{N}_0 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta}(1 + \mathcal{N}_a^*)
$$

$$
(\times)\frac{T^{2q\beta}}{q^2\beta^2} \sup_{0 \leq s \leq t} \mathbb{E}\|x(s) - \overline{x}(s)\|_{\mathcal{H}}^2, \quad \|\mathscr{A}^{-\beta}\|^2 = \mathcal{N}_0,
$$

$$
\leq \left[4\mathcal{M}^2 \left(\mathcal{N}_{\mathcal{A}_1} + \mathcal{N}_0 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2\beta^2} \right)(1 + \mathcal{N}_a^*) \right.
$$

$$
\left. + 4\mathcal{N}_{\mathcal{I}_p} + 4\mathcal{N}_{\overline{\mathcal{I}}_p} \right] \sup_{0 \leq s \leq t} \mathbb{E}\|x(s) - \overline{x}(s)\|_{\mathcal{H}}^2.
$$

Hence Φ_1 is a contraction mapping. $\qquad\square$

Lemma 9. *Let assumptions $(H1) - (H9)$ hold. Then Φ_2 maps bounded sets into itself in B_r.*

Proof. For all $t \in \mathscr{I}, x \in B_r$ and $\lambda > 0$, we have

$$
\mathbb{E}\|x_t + v_t\|_{\mathscr{B}}^2 \leq 4K_1^* r + C_1 = \widetilde{r}, \quad \mathbb{E}\|x_{\varrho(t, x_t + v_t)} + v_{\varrho(t, x_t + v_t)}\|^2 \leq 4K_1^* r + C_2 = r^*.
$$

$$
\mathbb{E}\|(\Phi_2 x)(t)\|_{\mathcal{H}}^2
$$

$$
\leq 4\mathbb{E}\left\| \int_0^t \int_0^s \mathcal{T}_q(t-s)\mathcal{A}_2(t-s)\mathcal{A}_1\left(\tau, x_\tau + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu + v_\mu)d\mu\right)d\tau ds \right\|_{\mathcal{H}}^2
$$

$$
+ 4\mathbb{E}\left\| \int_0^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds \right\|_{\mathcal{H}}^2 + 4\mathbb{E}\left\| \int_0^t \mathcal{T}_q(t-s) \right.
$$

$$
(\times)\left[\int_{-\infty}^s g(\tau, x_{\varrho(t, x_t + v_t)} + v_{\varrho(t, x_t + v_t)})dW(\tau) \right] ds \right\|_{\mathcal{H}}^2
$$

$$
+ 4\mathbb{E}\left\| \int_0^t \int_Z \mathcal{T}_q(t - t_p)g(s, x_s + v_s, z)\widetilde{N}(ds, dz) \right\|_{\mathcal{H}}^2
$$

$$
\leq 4\mathcal{M}^2 \mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta} \frac{T^{2q\beta}}{q^2\beta^2}\left(1 + \widetilde{r} + \mathcal{N}_a(1 + \widetilde{r})\right) \int_0^T \mu(s)ds + 4\left(\frac{11}{\beta^2}(\mathcal{M}\mathcal{M}_B)^4 T^2 \mathcal{N}_R \right)
$$

$$
+ 4\mathcal{M}^2 T^2 (2\mathcal{N}_k + 2Tr(Q)\mathcal{N}_h r^* \sup_{s \in \mathscr{I}} m(s)) + 4\mathcal{M}^2 T^2 (2\widetilde{\mathcal{N}}_g + 2\sqrt{L_g}) \widetilde{r} \sup_{s \in \mathscr{I}} n(s)
$$

$$
= \widetilde{\Delta}.
$$

Thus for each $x \in B_r, \mathbb{E}\|(\Phi_2 x)(t)\|_{\mathcal{H}}^2 \leq \widetilde{\Delta}$. $\qquad\square$

Lemma 10. *Let us assume $(H1) - (H9)$ hold. Then the set $\{\Phi_2 x : x \in B_r\}$ is an equicontinuous of family of functions on $\mathscr{I}$.*

Proof. Let $0 < \epsilon_1 < t < T$ and $\lambda_1 > 0$ such that $\|\mathcal{T}_q(s_1) - \mathcal{T}_q(s_2)\| < \epsilon_1$ with $|s_1 - s_2| < \lambda_1$ for $s_1, s_2 \in \mathscr{I}$. For $x \in B_r, 0 < |q_1| < \lambda_1, t + q_1 \in \mathscr{I}$, we have

$$\mathbb{E}\|\Phi_2 x(t + q_1) - \Phi_2 x(t)\|_{\mathcal{H}}^2$$

$$\leq 8\mathcal{M}^2\mathbb{E}\left\| \int_t^{t+q_1} \int_0^s \mathcal{A}_2(s-\tau)\mathcal{A}_1\left(\tau, x_\tau + v_\tau, \int_0^\tau a(\tau, \mu, x_\mu + v_\mu)d\mu\right)d\tau ds\right\|_{\mathcal{H}}^2$$

$$+ 8\mathbb{E}\left\| \int_0^t \int_0^s [\mathcal{T}_q(t + q_1 - s) - \mathcal{T}_q(t - s)]\mathcal{A}_2(s - \tau)\right.$$

$$(\times)\mathcal{A}_1\left(\tau, x_\tau + v_\tau, \int_0^s a(\tau, \mu, x_\mu + v_\mu)d\mu\right)d\tau ds\Big\|_{\mathcal{H}}^2$$

$$+ 8\mathcal{M}^2\mathbb{E}\left\| \int_t^{t+q_1} Bu^\lambda(s)ds\right\|_{\mathcal{H}}^2 + 8\mathbb{E}\left\| \int_0^t [\mathcal{T}_q(t + q_1 - s) - \mathcal{T}_q(t - s)]Bu^\lambda(s)ds\right\|_{\mathcal{H}}^2$$

$$+ 8\mathcal{M}^2\mathbb{E}\left\| \int_t^{t+q_1} \left[\int_{-\infty}^s g(\tau, x_{\varrho(\tau, x_\tau + v_\tau)} + v_{\varrho(\tau, x_\tau + v_\tau)})dW(\tau)\right]ds\right\|_{\mathcal{H}}^2$$

$$+ 8\mathbb{E}\left\| \int_0^t [\mathcal{T}_q(t + q_1 - s) - \mathcal{T}_q(t - s)]\right.$$

$$(\times)\left[\int_{-\infty}^s g(\tau, x_{\varrho(\tau, x_\tau + v_\tau)} + v_{\varrho(\tau, x_\tau + v_\tau)})dW(\tau)\right]ds\Big\|_{\mathcal{H}}^2$$

$$+ 8\mathcal{M}^2\mathbb{E}\left\| \int_t^{t+q_1} \mathcal{T}_q(t + q_1 - s)\left[\int_Z g(s, x_s + v_s, z)\widetilde{N}(ds, dz)\right]\right\|_{\mathcal{H}}^2$$

$$+ 8\mathbb{E}\left\| \int_0^t [\mathcal{T}_q(t + q_1 - s) - \mathcal{T}_q(t - s)] \int_Z g(s, x_s + v_s, z)\widetilde{N}(ds, dz)\right\|_{\mathcal{H}}^2$$

By using our assumptions $(H1) - (H9)$, we get

$$\mathbb{E}\|\Phi_2 x(t + q_1) - \Phi x(t)\|_{\mathcal{H}}^2$$

$$\leq 8\mathcal{M}^2\mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta}\frac{T^{q\beta}}{q\beta} \int_t^{t+q_1} (t + q_1 - s)^{q\beta-1}\widetilde{r}(1 + \mathcal{N}_a) \int_0^T \mu(s)dsd\tau$$

$$+ 8\epsilon_1^2\mathcal{N}_{\mathcal{A}_1, (-\mathscr{A})^\beta}\frac{T^{q\beta}}{q\beta} \int_0^t [(t + q_1 - s)^{q\beta-1} - (t - s)^{q\beta-1}]\widetilde{r}(1 + \mathcal{N}_a) \int_0^T \mu(s)dsd\tau$$

$$+ 8\mathcal{M}^2\mathcal{M}_B^2 \int_t^{t+q_1} \mathbb{E}\|u^\lambda(s)\|^2 ds + 8\epsilon_1^2\mathcal{M}_B^2 \int_0^t \mathbb{E}\|u^\lambda(s)\|^2 ds$$

$$+ 8\mathcal{M}^2 \int_t^{t+q_1} [2\mathcal{N}_k + 2Tr(Q)m(s)\mathcal{N}_h r^*]ds + 8\epsilon_1^2 \int_0^t [2\mathcal{N}_k + 2Tr(Q)m(s)\mathcal{N}_h r^*]ds$$

$$+ 8\mathcal{M}^2 \int_t^{t+q_1} (2\widetilde{\mathcal{N}}_g + 2\sqrt{L_g})n(s)\widetilde{r}ds + \epsilon_1^2 \int_0^t (2\widetilde{\mathcal{N}}_g + 2\sqrt{L_g})n(s)\widetilde{r}ds$$

Hence, for ϵ_1 sufficiently small, the right hand side of the above inequality $\to 0$ as q_1 tends to zero. On the other hand, the compactness of $\mathcal{T}_q(t), t > 0$ implies the continuity in the uniform operator topology. Thus, the set $\{\Phi_2 x : x \in B_r\}$ is equicontinuous family of functions. $\square$

Lemma 11. *Let us assume* $(H1) - (H9)$ *hold. Then* Φ_2 *maps* B_r *into a precompact set in* B_r.

Proof. Let $0 < t \le T$ be fixed and ϵ_1 be a real number that satisfy $0 < \epsilon_1 < t$. For $\lambda_1 > 0$, define the operator $\Phi_2^{\epsilon_1,\lambda_1}$ on B_r by

$$\Phi_2^{\epsilon_1,\lambda_1} = -\int_0^{t-\epsilon_1} \mathcal{T}_q(t-s)\mathcal{A}_2(s-\tau)\left[\int_0^s \mathcal{A}_1\left(\tau, x_t + v_t, \int_0^\tau a(\tau,\mu,x_\mu + v_\mu)d\mu\right)d\tau\right]ds$$
$$+ \int_0^{t-\epsilon_1} \mathcal{T}_q(t-s)Bu^\lambda(s)ds + \int_0^{t-\epsilon_1} \mathcal{T}_q(t-s)$$
$$(\times)\left[\int_{-\infty}^s g(\tau, x_{\varrho(\tau,x_\tau+v_\tau)} + v_{\varrho(\tau,x_\tau+v_\tau)})dW(\tau)\right]ds$$
$$+ \int_0^{t-\epsilon_1} \mathcal{T}_q(t-s)\left[\int_Z g(s, x_s + v_s, z)\widetilde{N}(ds,dz)\right].$$

But for $t > 0$, $\mathcal{T}_q(t)$ is a compact operator. Hence the set $\{(\Phi_2^{\epsilon_1,\lambda_1}x)(t) : x \in B_r\}$ is precompact in $\mathcal{H}$ for every $\epsilon_1 \in (0,t), \lambda_1 > 0$. Also for each $x \in B_r$, we have

$$\mathbb{E}\|(\Phi_2 x)(t) - (\Phi_2^{\epsilon_1,\lambda_1}x)(t)\|_\mathcal{H}^2$$
$$\le 4\epsilon_1^2\mathbb{E}\left\|\int_{t-\epsilon_1}^t \mathcal{T}_q(t-s)\mathcal{A}_2(s-\tau)\left[\int_0^s \mathcal{A}_1\left(\tau, x_t + v_t, \int_0^\tau a(\tau,\mu,x_\mu+v_\mu)d\mu\right)d\tau\right]ds\right\|_\mathcal{H}^2$$

$$+ 4\epsilon_1^2\mathbb{E}\left\|\int_{t-\epsilon_1}^t \mathcal{T}_q(t-s)Bu^\lambda(s)ds\right\|_\mathcal{H}^2$$
$$+ 4\epsilon_1^2\mathbb{E}\left\|\int_{t-\epsilon_1}^t \mathcal{T}_q(t-s)\left[\int_{-\infty}^s g(\tau, x_{\varrho(\tau,x_\tau+v_\tau)} + v_{\varrho(\tau,x_\tau+v_\tau)})dW(\tau)\right]ds\right\|_\mathcal{H}^2$$
$$+ 4\epsilon_1^2\mathbb{E}\left\|\int_{t-\epsilon_1}^t \mathcal{T}_q(t-s)\int_Z\left[g(s, x_s + v_s, z)\widetilde{N}(ds,dz)\right]ds\right\|_\mathcal{H}^2$$
$$\le 4\epsilon_1^2\mathcal{M}^2\int_{t-\epsilon_1}^t\left[\frac{T^{q\beta}}{q\beta}\mathcal{N}_{\mathcal{A}_1,(-\mathscr{A})^\beta}(t-s)^{q\beta-1}\left(1+\widetilde{r}(1+\mathcal{N}_a)\right)\int_0^T \mu(s)ds\right.$$
$$+ \mathcal{M}_B^2\frac{11}{\lambda^2}\mathcal{M}^2\mathcal{M}_B^2\mathcal{N}_R + 2\mathcal{N}_k + 2Tr(Q)m(s)\mathcal{N}_h r^*$$
$$+ 2(\widetilde{\mathcal{N}}_g + 2\sqrt{L_g})n(s)(1+\widetilde{r})\bigg]ds \to 0 \text{ as } \epsilon_1, \lambda_1 \to 0^-.$$

From the above inequality $\mathbb{E}\|(\Phi_2 x)(t) - (\Phi_2^{\epsilon_1,\lambda_1}x)(t)\|_\mathcal{H}^2$ tends to zero as $\epsilon_1, \lambda_1 \to 0^-$. Therefore, there are relatively compact sets arbitrarily close to the set $\{(\Phi_2 x)(t) : x \in B_r\}, t > 0$. Thus, the set $\Phi_2 x(t)$ is also precompact in B_r.

As a consequence of Lemmas 3.7-3.11 with Arezela - Ascoli's theorem, Φ satisifes all the conditions of Sadovskii's fixed point theorem on B_r. Which is the solution of fractional stochastic control system (1.1)-(1.5). Therefore, the problem (1.1)-(1.5) has a mild solution on $[0,T]$. $\square$

4. APPROXIMATE CONTROLLABILITY

Theorem 4.1. *Suppose that the assumptions of theorem 3.1 hold and the functions $\mathcal{A}_1, \mathcal{A}_2, h$ and g are uniformly bounded, and $\mathcal{R}_q(t)$ and $\mathcal{T}_q(t)$ are compact, then the stochastic integro-differential equations with state-dependent delay system (1.1)-(1.5) is approximately controllable on $\mathscr{I}$.*

Proof. Let x^λ be a fixed point of Φ^λ in B_r. By theorem 3.1 and the stochastic Fubini's theorem, we have

$$
\begin{aligned}
x^\lambda(T) = \overline{y}_T - \lambda(\lambda I + \Gamma_0^T)^{-1}\bigg[& \mathbb{E}\overline{y}_T + \int_0^T \overline{\varphi}(s)dW(s) - \mathcal{R}_q(T)\big[\varphi(0) + \mathcal{A}_1(0,\varphi,0) \\
& + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0)\big] + \mathcal{A}_1\Big(T, y_T^\lambda, \int_0^T a(T,s,y_s^\lambda)ds\Big)\bigg] \\
& - \lambda \int_0^T (\lambda I + \Gamma_0^T)^{-1}\mathscr{A}\mathcal{T}_q(T-s)\mathcal{A}_1\Big(s, y_s^\lambda, \int_0^s a(s,\tau,y_\tau^\lambda)d\tau\Big)ds \\
& - \lambda \int_0^T \int_0^s (\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)\mathcal{A}_2(T-s)\mathcal{A}_1\Big(\tau, y_\tau^\lambda, \int_0^\tau a(\tau,\mu,y_\mu^\lambda)d\mu\Big)d\tau ds \\
& + \lambda \int_0^T (\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)\int_{-\infty}^s h(\tau, y_{\varrho(\tau,y_\tau)}^\lambda)dW(\tau)ds \\
& + \lambda \int_0^T \int_Z (\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)g(s,y_s^\lambda,z)\widetilde{N}(ds,dz) \\
& + \lambda(\lambda I + \Gamma_0^T)^{-1} \sum_{0<t_p<t} \mathcal{T}_q(T-t_p)\mathcal{I}_p(y_{t_p}^\lambda) \\
& + \lambda(\lambda I + \Gamma_0^T)^{-1} \sum_{0<t_p<t} \mathcal{T}_q(T-t_p)\overline{\mathcal{I}}_p(y_{t_p}^\lambda).
\end{aligned}
$$

The properties of $\mathcal{A}_1, h$ and g implies that $\big\| h(\tau, y_{\varrho(\tau,y_\tau)}^\lambda) \big\|^2 + \big\| g(s,y_s^\lambda,z) \big\|^2 \leq \mathcal{M}_1$ and $\big\| \mathcal{A}_1\big(s, y_s^\lambda, \int_0^s a(s,\tau,y_\tau^\lambda)d\tau\big) \big\|_{\mathcal{H}}^2 \leq \mathcal{M}_2$. Also the properties of $\mathcal{I}_p$ and $\overline{\mathcal{I}}_p$ implies that $\|\mathcal{I}_p(y_{t_p}^\lambda)\| \leq \mathcal{M}_3$ and $\|\overline{\mathcal{I}}_p(y_{t_p}^\lambda)\|^2 \leq \mathcal{M}_4$. Then the subsequence

$$
\bigg\{ \mathcal{A}_1\Big(s, y_s^\lambda, \int_0^s a(s,\tau,y_\tau^\lambda)d\tau\Big), h(\tau, y_{\varrho(\xi,y_\xi)}^\lambda), g(s,y_s^\lambda), \mathcal{I}_p(y_{t_p}^\lambda), \overline{\mathcal{I}}_p(y_{t_p}^\lambda) \bigg\}
$$

converges weakly to, say, $\{\mathcal{A}_1(s), h(s,\tau), g(s,z), \mathcal{I}_p(y), \overline{\mathcal{I}}_p(y)\}$. Then, we have

$$
\begin{aligned}
\mathbb{E}\|x^\lambda(T) &- \overline{x}_T\|^2 \\
\leq 15 & \bigg\| \lambda(\lambda I + \Gamma_0^T)^{-1}[\mathbb{E}\overline{y}_T + \mathcal{R}_q(T)[\varphi(0) + \mathcal{A}_1(0,\varphi,0) + m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(0)]\bigg\|^2 \\
& + 15\mathbb{E}\bigg\| \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{A}_1\Big(T, y_T, \int_0^T a(T,s,y_s)ds\Big) \bigg\|^2 \\
& + 15\mathbb{E}\bigg(\int_0^T \big\| \lambda(\lambda I + \Gamma_0^T)^{-1}\varphi(s) \big\|_{\mathscr{L}_2^0}^2 ds \bigg) \\
& + 15\mathbb{E}\bigg(\int_0^T \big\| \lambda(\lambda I + \Gamma_0^T)^{-1}\mathscr{A}\mathcal{T}_q(T-s)\Big[\mathcal{A}_1\Big(s, y_s^\lambda, \int_0^s a(s,\tau,y_\xi^\lambda)d\tau\Big) - \mathcal{A}_1(s)\Big] \big\| ds \bigg)^2
\end{aligned}
$$

$$+ 15\mathbb{E}\left(\int_0^T \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathscr{A}\mathcal{T}_q(T-s)\mathcal{A}_1(s)\right\| ds\right)^2$$

$$+ 15\mathbb{E}\left(\int_0^T \int_0^s \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)\mathcal{A}_2(T-s)\right.\right.$$

$$(\times)\left[\mathcal{A}_1\left(s, y_s^\lambda, \int_0^s a(s, \tau, y_\xi^\lambda)d\tau\right) - \mathcal{A}_1(s)\right]\left.\Big\| ds\right)^2$$

$$+ 15\mathbb{E}\left(\int_0^T \int_0^s \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)\mathcal{A}_2(T-s)\mathcal{A}_1(s)\right\| ds\right)^2$$

$$+ 15\mathbb{E}\left(\int_0^T \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)\left[\int_{-\infty}^s [h(\tau, y_{\xi(\tau, y_\tau)}^\lambda) - h(s, \tau)]\right]\right\| ds\right)^2$$

$$+ 15\mathbb{E}\left(\int_0^T \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)\left[\int_{-\infty}^s h(s, \tau)\right]\right\| ds\right)^2$$

$$+ 15\mathbb{E}\left(\int_0^T \int_Z \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)[g(s, y_s^\lambda, z) - g(s, z)]\widetilde{N}(ds, dz)\right\| ds\right)^2$$

$$+ 15\mathbb{E}\left(\int_0^T \int_Z \left\|\lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-s)g(s, z)\widetilde{N}(ds, dz)\right\| ds\right)^2$$

$$+ 15\mathbb{E}\left\|\sum_{0<t_p<t} \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-t_p)[\mathcal{I}_p(y_{t_p}^\lambda) - \mathcal{I}_p(y)]\right\|^2$$

$$+ 15\mathbb{E}\left\|\sum_{0<t_p<t} \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-t_p)\mathcal{I}_p(y)\right\|^2$$

$$+ 15\mathbb{E}\left\|\sum_{0<t_p<t} \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-t_p)[\overline{\mathcal{I}}_p(y_{t_p}^\lambda) - \overline{\mathcal{I}}_p(y)]\right\|^2$$

$$+ 15\mathbb{E}\left\|\sum_{0<t_p<t} \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{T}_q(T-t_p)\overline{\mathcal{I}}_p(y)\right\|^2$$

Hence for all $0 \le s \le T$, the operator $\lambda(\lambda I + \Gamma_0^T)^{-1} \to 0$ strongly as $\lambda \to 0$ and moreover $\|\lambda(\lambda I + \Gamma_0^T)^{-1}\| \le 1$. Thus by the Lebesque dominated convergence theorem and the compactness of $\mathcal{R}_q(t)$ and $\mathcal{T}_q(t)$ implies that $\mathbb{E}\|x^\lambda(T) - \overline{x}_T\|^2 \to 0$. This gives the approximate controllability of the control system (1.1)-(1.5). $\square$

5. Example

Consider the folllowing fractional order partial differential system in the form

$$^C D_t^q\left[y(t, z) + \int_{-\infty}^t \int_0^\pi e_1(t-s, \overline{\mu}, z)y(s, \overline{\mu})d\overline{\mu}ds + \int_0^t \int_{-\infty}^s b_1(s-\tau, \overline{\mu}, z)y(\tau, \overline{\mu})d\overline{\mu}d\tau\right]$$

$$= \frac{\partial^2}{\partial z^2}y(t, z) + \beta(t, z) + \int_0^t (t-s)^\delta e^{-\gamma(t-s)}\frac{\partial^2}{\partial z^2}y(s, z)ds + \int_{-\infty}^t a_1(s-t)y(s, z)dW(s)$$

$$+ \int_Z \alpha\left(\int_{-\infty}^t \int_{-\infty}^s a_2(s-t)b_2(y(\tau, \varrho_1(\tau)\varrho_2(\|y(\tau)\|), z))y(\tau, z)ds\right)\widetilde{N}(dt, d\alpha),$$

$$(t, z) \in \mathscr{I} \times [0, \pi], \quad 0 \le t \le T, \quad 0 \le z \le \pi. \tag{5.1}$$

$$y(t,0) = y(t,\pi) = 0, \quad t \in \mathscr{I}, \tag{5.2}$$

$$y(\theta, z) = \varphi(\theta, z) + \sum_{i=0}^{m} C_i y(t_i + z), \quad \theta \leq 0, z \in [0, \pi], t \in (-\infty, 0], \tag{5.3}$$

$$[y(t_p^+) - y(t_p^-)]z = \mathcal{I}_p(y(t_p))z = \int_{-\infty}^{t} c_p(t_p - s)y(s, z)ds, \quad p = 1, 2, \cdots, n, \tag{5.4}$$

$$[y'(t_p^+) - y'(t_p^-)]z = \overline{\mathcal{I}}_p(y(t_p))z = \int_{-\infty}^{t} d_p(t_p - s)y(s, z)ds, \quad p = 1, 2, \cdots, n, \tag{5.5}$$

where $^C D_t^q$ is the Caputo fractional partial derivative of order $1 < q < 2$, $\beta \in \mathscr{I} \times [0, \pi] \to [0, \pi]$ is continuous. c_p, d_p are continuous for $p = 1, 2, \cdots, n$ and $C_i, i = 1, 2, \cdots, m$ are fixed numbers. Define the operator $m_1 : \mathscr{B}_h^m \to \mathscr{B}_h$ by $m_1(y_{t_1}, y_{t_2}, \cdots, y_{t_m})(z) = \sum_{i=0}^{m} C_i y(t_i + z)$ and $\|m_1(\cdot)\|_{\mathcal{H}}^2 \leq \mathcal{M}_{m_1}$. $\mathcal{I}_p$ and $\overline{\mathcal{I}}_p : \mathcal{H} \to \mathcal{H}$ are approximate functions. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < T$ be the given time points and the symbol $\Delta\zeta(t)$ represents the jump of the function ζ at t defined by $\Delta\zeta(t) = \zeta(t^+) - \zeta(t^-)$. Also, $W(t)$ denotes a standard one-dimensional Wiener process defined on a stochastic basis $(\Omega, \{\mathfrak{F}_t\}, \mathfrak{F}, \mathbb{P})$. To write the above system (5.1)-(5.5) into the abstract form (1.1)-(1.5), we can choose the space $\mathcal{H} = U = \mathscr{L}_2([0, \pi])$. $\mathscr{B}_h$ is the phase space. Define $\mathscr{A} : D(\mathscr{A}) \subseteq \mathcal{H} \to \mathcal{H}$ by $\mathscr{A}z = z''$, $(\mathscr{A}y)(z) = \frac{\partial^2 y(z)}{\partial z^2}$ with the domain $D(\mathscr{A}) = \{z \in \mathcal{H}, z, z' \text{ are absolutely continuous, } z'' \in \mathcal{H} \text{ and } z(0) = z(\pi) = 0\}$. The operator $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\phi(\mathscr{A}) = \mathbb{C}\{-n^2 : n \in \mathbb{N}\}$ and for $v \in (0,1)$ and $qv \in (\frac{\pi}{2}, \pi)$ there exists $\mathcal{M}_{qv} > 0$ such that $\|\mathbb{R}(\lambda, \mathscr{A})\| \leq \mathcal{M}_{qv}|\lambda|^{-1}$ for all $\lambda \in \sum_{qv}$ and the fractional power $(-\mathscr{A})^\gamma : D((-\mathscr{A}^\gamma)) \subset \mathcal{H} \to \mathcal{H}$ of $\mathscr{A}$ is given by $(-\mathscr{A})^\gamma z = \sum_{n=1}^{\infty} n^{2\gamma} \langle z, W_n \rangle W_n$, where $D((-\mathscr{A}^\gamma)) = \{z \in \mathcal{H} : (-\mathscr{A}^\gamma)z \in \mathcal{H}\}$. Hence, $\mathscr{A}$ is sectorial of type and the properties $(P1)$ hold. We also consider the operator $\mathcal{A}_2(t) : D(\mathscr{A}) \subseteq \mathcal{H} \to \mathcal{H}, t \geq 0, \mathcal{A}_2(t)z = t^\delta e^{-\gamma t}\mathscr{A}z$ for $z \in D(\mathscr{A})$. Moreover, it is easy to verify that conditions $(P2)$ and $(P3)$ are satisfied with $e_1(t) = t^\delta e^{-\gamma t}$ and $D = C_0^\infty([0, \pi])$ is the space of infinitely differentiable functions that vanish at $z = 0$ and $z = \pi$. Define the operators $\mathcal{A}_1 : \mathscr{I} \times \mathscr{B}_h \times \mathcal{H} \to \mathcal{H}, h : \mathscr{I} \times \mathscr{B}_h \to \mathcal{H}, g : \mathscr{I} \times \mathscr{B}_h \times Z \to \mathcal{H}, \varrho : \mathscr{I} \times \mathscr{B}_h \to \mathcal{H}$ and $\mathcal{I}_p, \overline{\mathcal{I}}_p : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}_1(\psi)(z) = \int_{-\infty}^{0} \int_{0}^{\pi} e_1(s, \overline{\mu}, z)\psi(s, \overline{\mu})d\overline{\mu}ds,$$

$$h(\psi)(z) = \int_{-\infty}^{0} a_1(s)\psi(s, x)dW(s),$$

$$g(\psi)(z) = \int_{-\infty}^{t} a_2(s)\psi(s, x)ds,$$

$$\mathcal{I}_p(\psi)(z) = \int_{-\infty}^{t} c_p(s)\psi(s, x)ds, \quad p = 1, 2, \cdots, n,$$

$$\overline{\mathcal{I}}_p(\psi)(z) = \int_{-\infty}^{t} d_p(s)\psi(s, x)ds, \quad p = 1, 2, \cdots, n.$$

Then the functions $\mathcal{A}_1, h$ and g are continuous. The bounded linear operator $B : U \to \mathcal{H}$ is defined by $Bu(t) = \beta(t, z), 0 \leq z \leq \pi$. Hence with the choices of [29],

the system (5.1)-(5.5) can be rewritten to the abstract form (1.1)-(1.5) and all the conditions of Theorem 3.1 are satisfied. Thus there exists a mild solution for the system (5.1)-(5.5). Moreover, all the conditions of Theorem 4.1 are satisfied and hence the system (5.1)-(5.5) is approximately controllable on $[0, T]$.

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