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THEOREM OF EXISTENCE AND UNIQUENESS OF SOLUTION FOR DIFFERENTIAL EQUATION OF FRACTIONAL ORDER

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Abstract. In this paper we proved a theorems of existence and uniqueness of solutions of differential equation of second order with fractional derivative in the Kipriyanov sense in lower terms. As a domain of definition of the functions we consider the n — dimensional Euclidean space. By a simple reduction of Kipriyanov operator to the operator of fractional differentiation in the sense of Marchaud these results can be considered valid for the operator of fractional differentiation in the sense of Riemann-Liouville, because of known fact coincidence of these operators on the classes of functions representable by the fractional integral.

1. Brief historical review

In 1960, the famous mathematician Kipriyanov I.A. in his paper [8] devoted to the properties of the eponymous operator formulated the theorem of existence and uniqueness of solutions for partial differential equations second order with operator fractional differentiation in the lower terms. It is noteworthy that the proof of this theorem did not publish. Mathematicians Djrbashian M.M., Nakhushev A.M. one of the first in their works researched the differential equation second order with fractional derivatives in the lower terms. In 1970 published the paper of Djrbashian M.M. [7], in which is probably the first time considered the problem of eigenvalues of the differential operator fractional order. In 1977 published the paper of Nakhushev A.M. [17]. The author was considering the differential operator second order with fractional derivatives in the sense of Riemann-Liouville in lower terms. In this paper proved the theorem, subsequently of great importance, establishing a relationship between the eigenvalues of homogeneous differential equation of second order with fractional derivative in lower terms and the zeros of functions Mittag-Leffler type. Investigations in this direction continued by Aleroev T.S. in 1982 published his paper [1] in which he establishing a relationship between the zeros of an entire function and eigenvalues of the boundary value problems for differential equations second order with fractional derivatives in the lower terms. Numerous research results of the last author published in the period from 1984 to 1994 in

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the series of papers [2]-[4]. Bangti Jean and William Randall in their paper [5] 2012 considered the inverse problem to the Sturm-Liouville problem for differential operator second order with fractional derivative in the lower terms. Starting from 2013 to 2017 by several authors published a papers devoted to a differential equations of fractional order: [21],[12],[?],[19],[13]. It remains to note that the theory of differential equations of fractional order is still relevant today.

2. Introduction

Accepting a notation [10] we assume that Ω is a convex domain of n — dimensional Euclidean space $\mathbb{E}^n$, P is a fixed point of the boundary $\partial\Omega$, $Q(r,\vec{\mathbf{e}})$ is an arbitrary point of Ω; we denote by $\vec{\mathbf{e}}$ as a unit vector having the direction from P to Q, using r for notation of Euclidean distance between points P and Q. We will consider classes of Lebesgue: $L_p(\Omega),\ 1 \le p < \infty$ complex valued functions. In polar coordinates summability f on Ω of degree p, means that

$$\int\limits_{\Omega} |f(Q)|^p dQ = \int\limits_{\omega} d\chi \int\limits_{0}^{d(\vec{\mathbf{e}})} |f(Q)|^p r^{n-1} dr < \infty,$$

where $d\chi$ is an element of the solid angle the surface of a unit sphere in $\mathbb{E}^n$ and ω is a surface of this sphere, $d := d(\vec{\mathbf{e}})$ is a length of the segment of ray going from point P in the direction $\vec{\mathbf{e}}$ within the domain Ω. Notation $\mathrm{Lip}\,\lambda,\ 0 < \lambda \le 1$ means the set of functions satisfying the Holder-Lipschitz condition

$$\mathrm{Lip}\,\lambda := \left\{\rho(Q):\ |\rho(Q) - \rho(P)| \le M r^{\lambda},\ P, Q \in \bar{\Omega}\right\}.$$

The operator of fractional differentiation in the sense of Kipriyanov defined in [8] by formal expression

$$\mathfrak{D}^{\alpha}(Q) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{0}^{r} \frac{[f(Q) - f(P + \vec{\mathbf{e}}t)]}{(r-t)^{\alpha+1}} \left(\frac{t}{r}\right)^{n-1} dt + C_n^{(\alpha)} f(Q) r^{-\alpha},\ P \in \partial\Omega,$$

$$C_n^{(\alpha)} = (n-1)!/\Gamma(n-\alpha),$$

according to theorem 2 [8] acting as follows

$$\mathfrak{D}^{\alpha} : \overset{0}{W_p^l}(\Omega) \to L_q(\Omega),\ lp \le n,\ 0 < \alpha < l - \frac{n}{p} + \frac{n}{q},\ p \le q < \frac{np}{n - lp}. \tag{1}$$

If in the condition (1) we have the strict inequality $q > p$, then for sufficiently small $\delta > 0$ the next inequality holds

$$\|\mathfrak{D}^{\alpha} f\|_{L_q(\Omega)} \le \frac{K}{\delta^{\nu}} \|f\|_{L_p(\Omega)} + \delta^{1-\nu} \|f\|_{L_p^l(\Omega)}, \tag{2}$$

where

$$\nu = \frac{n}{l}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{\alpha + \beta}{l}.$$

The constant K independents on $\delta,\ f$ and point $P \in \partial\Omega$; β is an arbitrarily small fixed positive number. Further we assume that $(0 < \alpha < 1)$. Denote $\mathrm{diam}\,\Omega = \mathfrak{d}$; $C, C_i = \mathrm{const},\ i \in \mathbb{N}_0$. We use for inner product of points $P = (P_1, P_2, ..., P_n)$ and $Q = (Q_1, Q_2, ..., Q_n)$ which belong to $\mathbb{E}^n$ a contracted notations $P \cdot Q = P^i \overline{Q_i} = \sum_{i=1}^n P_i \overline{Q_i}$, denote $|P - Q| = r$ as an Euclidean distance between P and Q. As usually denote $D_i u$ as a generalized derivative of the function u with respect to coordinate variable with index $(1 \le i \le n)$ and let

$Du=(D_1u,D_2u,...,D_nu)$. Denote $\vec{\mathrm{e}}_k$, $(1\leq k\leq n)$ as an ort on n — dimensional Euclidean space, and define the difference attitude $\triangle_k^h v=[v(Q+\vec{\mathrm{e}}_k h)-v(Q)]/h$. We will assume that all functions has a zero extension outside of $\bar{\Omega}$. Everywhere further, if not stated otherwise we will use the notations of [10], [8].

We define the familie of operators $\psi_{\varepsilon}^{-},\ \varepsilon>0$ as follows: $\mathrm{D}(\psi_{\varepsilon}^{-})\subset L_p(\Omega)$. In the right-side case

$$(\psi_{\varepsilon}^{-}f)(Q)=\begin{cases}\displaystyle\int\limits_{r+\varepsilon}^{d}\frac{f(P+\vec{\mathbf{e}}r)-f(P+\vec{\mathbf{e}}t)}{(t-r)^{\alpha+1}}dt,\ 0\leq r\leq d-\varepsilon,\\ \displaystyle\frac{f(Q)}{\alpha}\left(\frac{1}{\varepsilon^{\alpha}}-\frac{1}{(d-r)^{\alpha}}\right),\quad d-\varepsilon<r\leq d.\end{cases}$$

Following [20, p.181] we define a truncated fractional derivative similarly the derivative in the sense of Marchaud, in the right-side case

$$(\mathfrak{D}_{d-,\varepsilon}^{\alpha}f)(Q)=\frac{1}{\Gamma(1-\alpha)}f(Q)(d-r)^{-\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}(\psi_{\varepsilon}^{-}f)(Q).$$

Right-side fractional derivatives accordingly will be understood as a limit in the sense of norm $L_p(\Omega)$, $1\leq p<\infty$ of truncated fractional derivative

$$\mathfrak{D}_{d-}^{\alpha}f=\lim_{\substack{\varepsilon\to0\\(L_p)}}\mathfrak{D}_{d-,\varepsilon}^{\alpha}f.$$

Consider a boundary value problem for differential equation of fractional order, containing in the left side an uniformly elliptic operator with real-valued coefficients and fractional derivative in the sense of Kipriyanov in the lower terms

$$Lu:=-D_j(a^{ij}D_iu)+p\,\mathfrak{D}^{\alpha}u=f\in L_2(\Omega),\ \ i,j=\overline{1,n},\tag{3}$$

$$u\in H^2(\Omega)\cap H_0^1(\Omega),\tag{4}$$

$$a^{ij}(Q)\in C^1(\bar{\Omega}),\ a^{ij}\xi_i\xi_j\geq a_0|\xi|^2,\ a_0>0,\ p(Q)>0,\ p(Q)\in\mathrm{Lip}\,\lambda,\ \lambda>\alpha.\tag{5}$$

We will use a special case of the Green's formula

$$-\int\limits_{\Omega}v\,\overline{D_j(a^{ij}D_iu)}\,dQ=\int\limits_{\Omega}a^{ij}D_jv\,\overline{D_iu}\,dQ\,,\ u\in H^2(\Omega),v\in H_0^1(\Omega).\tag{6}$$

In later we will need a following lemma.

Lemma 1 Let $u,v\in L_2(\Omega)$, $\mathrm{dist}\,(\mathrm{supp}\,v,\,\partial\Omega)>2|h|$, then we have a following formula

$$\int\limits_{\Omega}\triangle_k^h v\,\overline{u}\,dQ=-\int\limits_{\Omega}v\,\triangle_k^{-h}\overline{u}\,dQ.\tag{7}$$

Proof. In assumptions of this lemma we have a following

$$\int\limits_{\Omega}\triangle_k^h v\,\overline{u}\,dQ=\frac{1}{h}\int\limits_{\Omega}[v(Q+e_kh)-v(Q)]\,\overline{u(Q)}\,dQ=$$

$$=\frac{1}{h}\int\limits_{\omega}d\chi\int\limits_{0}^{r}v(P'+\bar{\mathbf{e}}r)\,\overline{u(P'+\bar{\mathbf{e}}r-e_kh)}\,r^{n-1}dr-\frac{1}{h}\int\limits_{\Omega}v(Q)\,\overline{u(Q)}\,dQ=$$

$$=\frac{1}{h}\int\limits_{\Omega'}v(Q')\,\overline{u(Q'-e_kh)}\,dQ'-\frac{1}{h}\int\limits_{\Omega}v(Q)\,\overline{u(Q)}\,dQ,\ P'=P+e_kh,\ Q'=P'+\bar{\mathbf{e}}r,$$

where Ω' shift of the domain Ω on the distance h in the direction e_k. Note that in consequence of condition on the set: $\operatorname{supp} u$, we have: $\operatorname{supp} u_1 \subset \Omega \cap \Omega'$, $u_1(Q') = u(Q' - e_k h)$. Hence, finally we can rewrite the last relation as a following

$$\int\limits_{\Omega} \triangle_k^h v \, \overline{u} \, dQ = \frac{1}{h} \int\limits_{\Omega} v(Q) \overline{[u(Q - e_k h) - u(Q)]} \, dQ = - \int\limits_{\Omega} v \, \triangle_k^{-h} \overline{u} \, dQ.$$

The proof is complete.

The theorems of existence and uniqueness which will be proved in the next section based on the results obtained in the papers [15], [14].

3. Main theorems

Consider the boundary value problem (3),(4). The proved a strong accretive property for operators of fractional dierentiation provides the opportunity by using Lax-Milgram theorem to prove the theorem of existence and uniqueness of generalized solution for this problem.

Definition 1 We will call the element $z \in H_0^1(\Omega)$ as a generalized solution of the boundary value problem (3),(4) if the following integral identity holds

$$B(v, z) = (v, f)_{L_2(\Omega)}, \ \forall v \in H_0^1(\Omega), \tag{8}$$

where

$$B(v, u) = \int\limits_{\Omega} \left[a^{ij} D_j v \overline{D_i u} + (\mathfrak{D}^{\alpha}_{d-} p \, v) \, \overline{u} \right] \, dQ, \ u, v \in H_0^1(\Omega).$$

Theorem 1 There is an unique generalized solution of the boundary value problem (3),(4).

Proof. We will Show that the form (8) satisfies the conditions of Lax-Milgram theorem, particulary we will show that the next inequalities holds

$$|B(v, u)| \leq K_1 \|v\|_{H_0^1} \|u\|_{H_0^1}, \ \operatorname{Re} B(v, v) \geq K_2 \|v\|^2_{H_0^1}, \ u, v \in H_0^1(\Omega), \tag{9}$$

where $K_1 > 0, \ K_2 > 0$ are constants independents from real functions u, v.

Let us prove the first inequality of (9). Using the Cauchy-Schwarz inequality for a sum, we have

$$a^{ij} D_j v \overline{D_i u} \leq a(Q) |Dv| |Du|, \ a(Q) = \left(\sum_{i,j=1}^{n} |a_{ij}(Q)|^2 \right)^{1/2}.$$

Hence

$$\left| \int\limits_{\Omega} a^{ij} D_j v \overline{D_i u} \, dQ \right| \leq P \|v\|_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)}, \ P = \sup_{Q \in \Omega} |a(Q)|. \tag{10}$$

In consequence of lemma 1 [15], lemma 2 [15], we have

$$(\mathfrak{D}^{\alpha}_{d-} p \, v, u)_{L_2(\Omega)} = (v, \mathfrak{D}^{\alpha} u)_{L_2(\Omega, p)}, \ u, v \in H_0^1(\Omega). \tag{11}$$

Applying the inequality (2), then Jung's inequality we get

$$\left| (v, \mathfrak{D}^{\alpha} u)_{L_2(\Omega, p)} \right| \leq C_0 \|v\|_{L_2(\Omega)} \|\mathfrak{D}^{\alpha} u\|_{L_q(\Omega)} \leq$$

$$\leq C_0 \|v\|_{L_2(\Omega)} \left\{ \frac{K}{\delta^{\nu}} \|u\|_{L_2(\Omega)} + \delta^{1-\nu} \|u\|_{L_2^1(\Omega)} \right\} \leq$$

$$\leq \frac{1}{\varepsilon}\|v\|^2_{L_2(\Omega)} + \varepsilon\left(\frac{KC_0}{\sqrt{2}\delta^{\nu}}\right)^2\|u\|^2_{L_2(\Omega)} + \frac{\varepsilon}{2}\left(C_0\delta^{1-\nu}\right)^2\|u\|^2_{L^1_2(\Omega)},$$

$$2 < q < \frac{2n}{2\alpha - 2 + n},\ C_0 = (\mathrm{mess}\,\Omega)^{\frac{q-2}{q}}\sup_{Q\in\Omega} p(Q).$$

Applying the Friedrichs inequality, finally we have a following estimate

$$|(\mathfrak{D}^{\alpha}_{d-}p\,v, u)_{L_2(\Omega)}| \leq C\|v\|_{H^1_0}\|u\|_{H^1_0}. \tag{12}$$

Note that from inequalities (10),(12) follows the first inequality of (9). Using the inequalities (28) [15], (36) [15], we have

$$\mathrm{Re}\,B(v,v) \geq a_0\|v\|^2_{L^1_2(\Omega)} + \lambda^{-2}\|v\|^2_{L_2(\Omega,p)} \geq$$

$$\geq a_0\|v\|^2_{L^1_2(\Omega)} + \lambda^{-2}p_0\|v\|^2_{L_2(\Omega)},\ p_0 = \inf_{Q\in\Omega} p(Q). \tag{13}$$

It is obviously that

$$a_0\|v\|^2_{L^1_2(\Omega)} + \lambda^{-2}p_0\|v\|^2_{L_2(\Omega)} \geq K_2\left(\|v\|^2_{L^1_2(\Omega)} + \|v\|^2_{L_2(\Omega)}\right) =$$

$$= K_2\left(\int\limits_{\Omega}\sum_{i=1}^{n}|D_i v|^2 dQ + \int\limits_{\Omega}|v|^2 dQ\right) = K_2\|v\|^2_{H^1_0},\ K_2 = \min\{a_0, \lambda^{-2}p_0\}. \tag{14}$$

Hence the second inequality of (9) follows from the inequalities (3), (14).

Since conditions of Lax-Milgram theorem holds, then for all bounded on $H^1_0(\Omega)$ functional F, exist unique element $z \in H^1_0(\Omega)$ such as

$$B(v,z) = F(v),\ \forall v \in H^1_0(\Omega). \tag{15}$$

Consider the functional

$$F(v) = (v,f)_{L_2(\Omega)},\ f \in L_2(\Omega),\ v \in H^1_0(\Omega). \tag{16}$$

Applying the Cauchy-Schwarz inequality, we get

$$|F(v)| = |(v,f)_{L_2(\Omega)}| \leq \|f\|_{L_2(\Omega)}\|v\|_{H^1_0(\Omega)}.$$

Hence the functional (16) is bounded on $H^1_0(\Omega)$, then in accordance with (15) we have equality

$$B(v,z) = (v,f)_{L_2(\Omega)},\ \forall v \in H^1_0(\Omega). \tag{17}$$

Therefore in accordance with definition 1 element z is an unique generalized solution of the boundary value problem (3),(4). The proof is complete.

The theorem 1 allows to prove the theorem of existence and uniqueness of solution of the boundary value problem (3),(4).

Theorem 2 There is an unique strong solution of the boundary value problem (3),(4).

Proof. In consequence of theorem 1 exists unique element $z \in H^1_0(\Omega)$, so that equality (17) is true. Note that if a generalized solution of the boundary value problem (3),(4) belongs to a Sobolev space $H^2(\Omega)$, then applying formulas (6),(11) we get

$$(v, Lz)_{L_2(\Omega)} = B(v,z) = (v,f)_{L_2(\Omega)},\ \forall v \in C^{\infty}_0(\Omega), \tag{18}$$

hence

$$(v, Lz - f)_{L_2(\Omega)} = 0,\ \forall v \in C^{\infty}_0(\Omega). \tag{19}$$

Since it is well known that there is no non-zero element in the Hilbert space which is orthogonal to the dense manifold, then z is solution of the boundary value problem(3),(4).

Let's prove that $z \in H^2(\Omega)$. Choose the function v in (17) so that $\overline{(\operatorname{supp} v)} \subset \Omega$, performing easy calculation, using equality (11), we get

$$\int\limits_{\Omega} a^{ij} D_j v \overline{D_i z}\, dQ = \int\limits_{\Omega} v\overline{q}\, dQ,\ \forall v \in H_0^1(\Omega),\ \overline{(\operatorname{supp} v)} \subset \Omega, \tag{20}$$

where $q = (f - p\,\mathfrak{D}^{\alpha} z)$. In the last equality for $2|h| < \operatorname{dist}(\operatorname{supp} v,\ \partial\Omega)$, let's change the function v on it difference attitude $\triangle^{-h} v = \triangle_k^{-h} v$ for some $1 \le k \le n$, then applying lemma 1 we get

$$-\int\limits_{\Omega} a^{ij} (D_j \triangle^{-h} v) \overline{D_i z}\, dQ = -\int\limits_{\Omega} (\triangle^{-h} v)\, \overline{q}\, dQ =$$

$$= -\int\limits_{\Omega} (D_j \triangle^{-h} v) \overline{a^{ij} D_i z}\, dQ = \int\limits_{\Omega} D_j v\, \overline{\triangle^h \left(a^{ij} D_i z\right)} dQ.$$

Using elementary calculation we get

$$\triangle^h \left(a^{ij} D_i z\right)(Q) = a^{ij}(Q + h\,\vec{e}_k)(D_i \triangle^h z)(Q) + [\triangle^h a^{ij}(Q)](D_i z)(Q),$$

hence

$$\int\limits_{\Omega} D_j v\, \overline{a^{ij}(Q + h\,\vec{e}_k)(D_i \triangle^h z)}\, dQ = -\int\limits_{\Omega} Dv \cdot g + (\triangle^{-h} v)\, \overline{q}\, dQ,$$

where $g = (g_1, g_2, ..., g_n),\ g_j = (\triangle^h a^{ij}) D_i z$. Note a last relation, using the Cauchy Schwarz inequality, finiteness property of function v, lemma 7.23 [6, p.164] we have

$$\left| \int\limits_{\Omega} a^{ij}(Q + h\,\vec{e}_k) D_j v\, \overline{(D_i \triangle^h z)}\, dQ \right| = \left| \int\limits_{\Omega} D_j v\, \overline{a^{ij}(Q + h\,\vec{e}_k)(D_i \triangle^h z)}\, dQ \right| \le$$

$$\le \|Dv\|_{L_2(\Omega)} \|g\|_{L_2(\Omega)} + \|\triangle^{-h} v\|_{L_2(\Omega)} \|q\|_{L_2(\Omega)} \le$$

$$\le \|Dv\|_{L_2(\Omega)} \left(\|g\|_{L_2(\Omega)} + \|q\|_{L_2(\Omega)} \right). \tag{21}$$

Applying the Cauchy Schwarz inequality for finite sum and integrals, it is easy to see that

$$\|g\|_{L_2(\Omega)} = \left(\int\limits_{\Omega} \sum_{j=1}^{n} |(\triangle^h a^{ij}) D_i z|^2 dQ \right)^{1/2} \le \left(\int\limits_{\Omega} |Dz|^2 \sum_{i,j=1}^{n} |\triangle^h a^{ij}|^2 dQ \right)^{1/2} \le$$

$$\le \sup_{Q \in \Omega} \left(\sum_{i,j=1}^{n} \left| \triangle^h a^{ij}(Q) \right|^2 \right)^{1/2} \left(\int\limits_{\Omega} |Dz|^2 dQ \right)^{1/2} \le C_1 \|z\|_{H^1(\Omega)}.$$

Note that using (2), we have

$$\|q\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)} + \|p\,\mathfrak{D}^{\alpha} z\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)} + C_2 \|z\|_{H^1(\Omega)}.$$

Using given above, from (3) we get

$$\left| \int\limits_{\Omega} a^{ij}(Q + h\,\vec{e}_k) D_j v\, \overline{(D_i \triangle^h z)}\, dQ \right| \le C \left(\|z\|_{H^1(\Omega)} + \|f\|_{L_2(\Omega)} \right) \|Dv\|_{L_2(\Omega)}. \tag{22}$$

Note that using condition (5), we can get a following estimate

$$\left|\int_{\Omega} a^{ij}\xi_j\overline{\xi_i}dQ\right| = \left|\int_{\Omega} a^{ij}(\mathrm{Re}\xi_j\mathrm{Re}\xi_i+\mathrm{Im}\xi_j\mathrm{Im}\xi_i)dQ+\imath\int_{\Omega} a^{ij}(\mathrm{Re}\xi_i\mathrm{Im}\xi_j-\mathrm{Re}\xi_j\mathrm{Im}\xi_i)dQ\right| =$$

$$= \left\{\left(\int_{\Omega} a^{ij}(\mathrm{Re}\xi_j\mathrm{Re}\xi_i+\mathrm{Im}\xi_j\mathrm{Im}\xi_i)dQ\right)^2 + \left(\int_{\Omega} a^{ij}(\mathrm{Re}\xi_i\mathrm{Im}\xi_j-\mathrm{Re}\xi_j\mathrm{Im}\xi_i)dQ\right)^2\right\}^{1/2} \geq$$

$$\geq \int_{\Omega} a^{ij}(\mathrm{Re}\xi_j\,\mathrm{Re}\xi_i+\mathrm{Im}\xi_j\,\mathrm{Im}\xi_i)\,dQ \geq k_0\int_{\Omega}|\xi|^2\;dQ. \tag{23}$$

Define the function χ, so that: $\mathrm{dist}\,(\mathrm{supp}\,\chi,\,\partial\Omega) > 2|h|$,

$$\chi(Q) = \begin{cases} 1, & Q\in\mathrm{supp}\,\chi, \\ 0, & Q\in\bar{\Omega}\setminus\mathrm{supp}\,\chi. \end{cases}$$

Suppose that $v=\chi\triangle^h z$. Using relations (22), (3), we have two-sided estimate

$$k_0\|\chi\triangle^h Dz\|^2_{L_2(\Omega)} \leq \left|\int_{\Omega}\chi a^{ij}(Q+h\,\vec{\mathrm{e}}_k)\triangle^h D_j z\overline{\triangle^h D_i z}\,dQ\right| =$$

$$= \left|\int_{\Omega} a^{ij}(Q+h\,\vec{\mathrm{e}}_k)D_j(\chi\triangle^h z)\,\overline{(D_i\triangle^h z)}\,dQ\right| \leq$$

$$\leq C\left(\|z\|_{H^1(\Omega)}+\|f\|_{L_2(\Omega)}\right)\|\chi\triangle^h Dz\|_{L_2(\Omega)}. \tag{24}$$

Using the Jung's inequality, for all positive k we get an estimate

$$2\left(\|z\|_{H^1(\Omega)}+\|f\|_{L_2(\Omega)}\right)\|\chi\triangle^h Dz\|_{L_2(\Omega)} \leq$$

$$\leq \frac{1}{k}\left(\|z\|_{H^1(\Omega)}+\|f\|_{L_2(\Omega)}\right)^2 + k\|\chi\triangle^h Dz\|^2_{L_2(\Omega)}.$$

Choosing $k<2k_0C^{-1}$, we can perform inequality (3) as follows

$$\|\chi\triangle^h Dz\|^2_{L_2(\Omega)} \leq C_1\left(\|z\|_{H^1(\Omega)}+\|f\|_{L_2(\Omega)}\right)^2.$$

It implies that for all domain Ω', $\mathrm{dist}(\Omega',\partial\Omega)>2|h|$, we have

$$\|\triangle_i^h D_j z\|_{L_2(\Omega')} \leq C_2\left(\|z\|_{H^1(\Omega)}+\|f\|_{L_2(\Omega)}\right),\; i,j=1,2,...,n.$$

In consequence of lemma 7.24 [6, p.165], we have that exists generalized derivative D_iD_jz and satisfies the condition

$$\|D_iD_jz\|_{L_2(\Omega)} \leq C_2\left(\|z\|_{H^1(\Omega)}+\|f\|_{L_2(\Omega)}\right),\; i,j=1,2,...,n.$$

Hence $z\in H^2(\Omega)$. The proof is complete.

4. Conclusions

We were guided by a well known, in the classical case of equations with positive-integer orders, idea of connection between the solvability of a boundary-value problem and a properties of the corresponding quadratic functional. A project to use the same approach in the fractional case required a some technique of the fractional calculus theory, in particular we used a strong accretive property of the fractional differentiation operators. Applying the Lax-Milgram theorem we proved the existence of a generalized solution of the boundary value problem for differential equation of fractional order. Was proved the inclusion of a generalized solution in the Sobolev class of functions, corresponding to the strong solution. Although this method also is not new in the theory of partial differential equations, it should be noted that in the proofs of the theorems was used a new technique of fractional calculus theory. The result is proved for multidimensional operator which has a reduction to various operators of fractional order.

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