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IMPULSIVE STOCHASTIC FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH STATE-DEPENDENT DELAY

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Abstract. This paper deals with the existence of mild solutions for impulsive fractional order stochastic integro-differential inclusions with state-dependent delay. The existence result is obtained by using a fixed point technique on a Hilbert space. An illustrating example is presented.

1. Introduction

Fractional-order models are found to be more adequate than integer-order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc., involves derivatives of fractional order. With regard to this matter, we refer the reader to [1, 2, 7, 13, 20, 23, 25, 33, 35].

The theory of impulsive differential equations of integer order has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [3, 4, 5, 6, 8, 9, 24, 32, 34].

However, on the one hand, there has been an increasing interest in extending certain classical deterministic results to stochastic differential equations. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic. Stochastic differential equations arise naturally in characterizing many problems in physics, biology, mechanics and so on; see [10, 15, 29] and the references therein. Recently, Lin and Hu [27] proved existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions, whereas Guendouzi

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and Benzatout [16] investigated the existence of solutions for fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators. In [36], Yan and Zhang studied the existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay.

Motivated by the above literature, the aim of this work is to establish the existence of mild solutions for a class of impulsive fractional stochastic integro-differential inclusions with state-dependent delay in a Hilbert space described by the form

$$\begin{aligned} {}^{C}D_t^q x(t) &\in Ax(t) + \int_0^t a(t,s)F(s, x_{\rho(s,x_s)}, x(s))dw(s),\ t \in J_k = (t_k, t_{k+1}], k = 0, 1, \ldots, m, \\ \Delta x(t_k) &= I_k(x(t_k^-)), \qquad k = 1, 2, \ldots, m, \\ x_0 &= \phi \in \mathcal{B}, \qquad t \in\ (-\infty, 0], \end{aligned} \tag{1.1}$$

where ${}^{C}D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$. The operator A generates a compact and uniformly bounded linear semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $(H, \|\cdot\|)$. The time history $x_t : (-\infty, 0] \to H$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. The function $F : [0, T] \times \mathcal{B} \times H \longrightarrow \mathcal{P}(L(K, H))$ is a multivalued map, $\rho : J \times \mathcal{B} \to (-\infty, T]$, $a : D \to \mathbb{R},\ (D = \{(t, s) \in [0, T] \times [0, T] : t \geq s\})$. Let K be another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. The initial data $\phi = \{\phi(t),\ t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment. Here, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, I_k : H \to H,\ k = 1, 2, \ldots, m$, are given maps, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ denote the right and the left limit of $x(t)$ at $t = t_k$, respectively, and $\phi \in \mathcal{B}$.

2. Preliminaries

In this section, we introduce some basic definitions, notation and lemmas which are used throughout this paper.

Let H and K be two separable Hilbert spaces and $L(K, H)$ denotes the space of all bounded linear operators from K into H. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in H, K and $L(K, H)$, and use $(\cdot,\cdot)$ to denote the inner product of H and K without any confusion. Let $(\Omega, \mathcal{F}, P)$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Let $w = (w_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, P)$ with the covariance operator Q such that $TrQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, \ldots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t), e)_K = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_K \beta_k(t), e \in K, t \geq 0.$$

Let $\psi \in L(K, H)$ and define

$$\|\psi\|_Q^2 = Tr(\psi Q \psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2.$$

If $\|\psi\| <?\infty$, then ψ is called a Q-Hilbert Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators ψ. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology. The collection of all strongly measurable, square integrable, H-valued random variables, denoted by $L_2(\Omega, H)$ is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot, w)\|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $Ex = \int_\Omega x(w)dP$. Let $C(J, L_2(\Omega, H))$ be the Banach space of all continuous maps from J into $L_2(\Omega, H)$ satisfying the condition $\sup_{0\leq t\leq T} E\|x(t)\|^2 < \infty$. Let $L_2^0(\Omega, H)$ denote the family of all $\mathcal{F}_0$-measurable, H-valued random variables $x(0)$.

$B_r(x, H)$ represents the closed ball in H with the center at x and the radius r.

Denote by $\mathcal{P}(H)$ the family of all nonempty subsets of H. Let $\mathcal{P}_{cl}(H) = \{Y \in \mathcal{P}(H) : Y \text{ closed}\}$, $\mathcal{P}_b(H) = \{Y \in \mathcal{P}(H) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(H) = \{Y \in \mathcal{P}(H) : Y \text{ compact}\}$, $\mathcal{P}_{cp,c}(H) = \{Y \in \mathcal{P}(H) : Y \text{ compact, convex}\}$, $\mathcal{P}_{cd}(H) = \{Y \in \mathcal{P}(H) : Y \text{ compact-acyclic }\}$.

A multivalued map $G : H \to \mathcal{P}(H)$ is convex (closed) valued if $G(H)$ is convex (closed) for all $x \in H$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in H for all $B \in \mathcal{P}_b(H)$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on H if for each $x_0 \in H$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set U of H containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathcal{P}_b(H)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$).
For more details on multivalued maps see the books of Deimling [12], Górniewicz [14] and Hu and Papageorgiou [22].

Definition 2.1. *([10]). We call $S \subset \Omega$ a P-null set if there is $B \in \mathcal{F}$ such that $S \subseteq B$ and $P(B) = 0$.*

Definition 2.2. *([10]). A stochastic process $\{x(t) : t \geq 0\}$ in a real separable Hilbert space H is a Wiener process if for each $t \geq 0$,*

- **:** *(i) $x(t)$ has continuous sample paths and independent increments.*
- **:** *(ii) $x(t) \in L^2(\Omega, H)$ and $E(x(t)) = 0$.*
- **:** *(iii) $Cov(w(t) - w(s)) = (t - s)Q$, where $Q \in L(K, H)$ is a nonnegative nuclear operator.*

Definition 2.3. *([10]). Brownian motion is a continuous adapted real-valued process $x(t), t \geq 0$ such that*

- **:** *(i) $x(0) = 0$.*
- **:** *(ii) $x(t) - x(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$.*
- **:** *(iii) $x(t) - x(s)$ is $N(0, t - s)$-distributed for all $0 \leq s \leq t$.*

Definition 2.4. *([10]). The process x is $\mathcal{F}_0$-adapted if each $x(0)$ is measurable with respect to $\mathcal{F}_0$.*

Definition 2.5. *Let $\alpha > 0$ and $f : \mathbb{R}_+ \to E$ be in $L^1(\mathbb{R}_+, E)$. Then the Riemann-Liouville integral is given by:*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds.$$

For more details on the Riemann-Liouville fractional derivative, we refer the reader to [11].

Definition 2.6. *([31]). The Caputo derivative of order α for a function f : $[0, +\infty) \to \mathbb{R}$ can be written as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 \leq \alpha < n.$$

If $0 \leq \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

Obviously, The Caputo derivative of a constant is equal to zero.

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [17]. Specifically, $\mathcal{B}$ will be a linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into H endowed with a seminorm $\|\cdot\|_\mathcal{B}$, and satisfies the following axioms:

(A1): If $x : \ (-\infty, T\ ] \longrightarrow H$ is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, $x_t \in \mathcal{B}$ and

$$\|x(t)\| \leq C\|x_t\|_\mathcal{B}, \tag{2.1}$$

where $C > 0$ is a constant.

(A2): There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_\mathcal{B} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_\mathcal{B}, \tag{2.2}$$

for $t \in [0, T]$ and x as in $(A1)$.

(A3): The space $\mathcal{B}$ is complete.

Remark 2.7. *Condition (2.1) in (A1) is equivalent to $\|\phi(0)\| \leq C\|\phi\|_\mathcal{B}$, for all $\phi \in \mathcal{B}$.*

Example 2.8. ***The phase space*** $C_r \times L^p(g, X)$.

Let $r \geq 0, 1 \leq p < \infty$, and let $g : (-\infty, -r) \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions $(g-5), (g-6)$ in the terminology of [21]. Briefly, this means that g is locally integrable and there exists a nonnegative, locally bounded function Λ on $(-\infty, 0]$, such that $g(\xi + \theta) \leq \Lambda(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r)\backslash N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$, such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|\cdot\|_\mathcal{B}$ is defined by

$$\|\varphi\|_\mathcal{B} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

*The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms **(A1), (A2), (A3)**. Moreover, for $r = 0$ and $p = 2$, this space coincides with $C_0 \times L^2(g, X), H = 1, M(t) = \Lambda(-t)^{\frac{1}{2}}, K(t) = 1 + \left(\int_{-r}^{0} g(\tau)d\tau\right)^{\frac{1}{2}}$, for $t \geq 0$ (see [21], Theorem 1.3.8 for details).*

The next result is a consequence of the phase space axioms.

Lemma 2.9. *If $x : (-\infty, T] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \phi(t) \in L^2_0(\Omega, \mathcal{B})$, then*

$$\|x_s\|_{\mathcal{B}} \leq C_2^* E\|\phi\|_{\mathcal{B}} + C_1^* \sup_{0 \leq s \leq T} E\|x(s)\|,$$

where $C_1^* = \sup\{C_1(t) : 0 \leq t \leq T\}, C_2^* = \sup\{C_2(t) : 0 \leq t \leq T\}$.

Definition 2.10. *The multivalued map $F : J \times \mathcal{B} \times H \longrightarrow \mathcal{P}(L(K, H))$ is said to be Carathéodory if*

(i) *$t \longmapsto F(t, x, y)$ is measurable for each $(x, y) \in \mathcal{B} \times H$;*
(ii) *$(x, y) \longmapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in J$.*

Definition 2.11. *Let $G : H \to \mathcal{P}_{bd,cl}(H)$ be a multi-valued map. Then G is called a multi-valued contraction if there exists a constant $\gamma \in [0, 1)$ such that for each $x, y \in H$ we have*

$$H_d(G(x), G(y)) \leq \gamma\|x - y\|.$$

The constant γ is called a contraction constant of G.

Lemma 2.12. *([18, 37]). Suppose $b \geq 0, \alpha > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (for some $T \leq +?\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b?\int_0^t (t - s)^{\alpha - 1} u(s)ds$$

on this interval; then

$$u(t) \leq a(t) +?\int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t - s)^{n\alpha - 1} a(s)\right] ds.$$

Let $S_{F,x}$ be a set defined by

$$S_{F,x} = \{v \in L^2(J, L(K, H)) : v(t) \in F(t, x_{\rho(t,x_t)}, x(t)) \text{ a.e. } t \in J\}.$$

Lemma 2.13. *([26]). Let H be a Hilbert space. Let $F : J \times \mathcal{B} \times H \longrightarrow P_{cp,c}(L(K, H))$ be an L^2-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^2(J, H)$ to $C(J, H)$, then the operator*

$$\begin{aligned} \Psi \circ S_F : C(J, H) &\longrightarrow P_{cp,c}(C(J, H)), \\ x &\longmapsto (\Psi \circ S_F)(x) := \Psi(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C(J, H) \times C(J, H)$.

Now we have a nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan.

Theorem 2.14. *([30]). Let H be a Hilbert space with V an open, convex subset of H and $y_0 \in H$. Suppose the following hold:*

(a): *$\Phi : \overline{V} \to P_{cd}(H)$ has closed graph, and*

(b): *$\Phi : \overline{V} \to P_{cd}(H)$ is a condensing map with $\Phi(\overline{V})$ a subset of a bounded set in H.*

Then either

(i): *Φ has a fixed point in $\overline{V}$, or*

(ii): *There exist $y \in \partial V$ and $\lambda \in (0,1)$ with $y \in \lambda\Phi(y) + (1-\lambda)\{y_0\}$.*

Now we consider the space

$$\mathcal{D}_T = \Big\{ x : (-\infty, T] \to H \textit{ such that } x|_{J_k} \in C(J_k, H) \textit{ and there exist } x(t_k^+) \textit{ and } x(t_k^-) \textit{ with } x(t_k) = x(t_k^-),\ x_0 = \phi, k = 1, 2, \dots, m \Big\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}], k = 1, 2, \dots, m$. Set $\|\cdot\|_T$ to be a seminorm in $\mathcal{D}_T$ defined by

$$\|x\|_{\mathcal{D}_T} = \sup_{s\in[0,T]} E(\|x(s)\|^2)^{\frac{1}{2}} + \|\phi\|_{\mathcal{B}},\ x \in \mathcal{D}_T.$$

3. Main results

In this section we shall present and prove our main result.

Definition 3.1. *An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to H$ is called a mild solution of system (1.1) if the following hold: $x_0 = \phi \in \mathcal{B}$ on $(-\infty, ?0], \Delta x(t_k) = I_k(x(t_k^-)), k = 1, 2, \dots, m$, the restriction of $x(\cdot)$ to the interval $J_k, (k = 0, 1, \dots, m)$ is continuous and there exists $v(\cdot) \in L^2(J_k, L(K, H))$, such that $v(t) \in F(t, x_{\rho(t,x_t)}, x(t))$ a.e. $t \in [0, T]$, and $x(t)$ satisfies the following fractional integral equation*

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)v(\tau)dw(\tau)ds & \\ + \sum_{0<t_k<t} Q(t-t_k)I_k(x(t_k^-)), & t \in J, \end{cases} \tag{3.1}$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma)S(t^q\sigma)d\sigma, \qquad R(t) = q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)d\sigma$$

and for $\sigma \in (0, \infty)$,

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0,$$

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q).$$

Here, ξ_q is a probability density function defined on $(0, \infty)$ [28], that is,

$$\xi_q(\sigma) \geq 0, \quad \sigma \in (0, \infty) \quad \textit{and} \quad \int_0^\infty \xi_q(\sigma)d\sigma = 1.$$

It is not difficult to verify that

$$\int_0^\infty \sigma\xi_q(\sigma)d\sigma = \frac{1}{\Gamma(1+q)}.$$

Remark 3.2. *Note that* $\{S(t)\}_{t\geq 0}$ *is a uniformly bounded semigroup, i.e.,*

$$\textit{there exists a constant} \quad M>0 \quad \textit{such that} \quad \|S(t)\| \leq M \quad \textit{for all} \quad t\in[0,T].$$

Remark 3.3. *Note that*

$$\|R(t)\| \leq C_{q,M} t^{q-1}, \quad t>0, \tag{3.2}$$

where $C_{q,M} = \dfrac{qM}{\Gamma(1+q)}$.

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J\times\mathcal{B}, \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J\times\mathcal{B} \to (-\infty, T]$ is continuous. Additionally, we introduce the following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t\in\mathcal{R}(\rho^-).$$

Remark 3.4. *The condition* (H_φ)*, is frequently satisfied by functions continuous and bounded. For more details, see for instance* [21].

Lemma 3.5. [19] *If* $x : (-\infty, T] \to H$ *is a function such that* $x_0 = \phi$*, then*

$$\|x_s\|_{\mathcal{B}} \leq (C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* \sup\{|y(\theta)|; \theta\in[0, max\{0,s\}]\}, \;\; s\in\mathcal{R}(\rho^-)\cup J,$$

where $L^\phi = \sup\limits_{t\in\mathcal{R}(\rho^-)} L^\phi(t)$.

Further we impose the following conditions.

(H1) The multivalued map $F : J\times\mathcal{B}\times X \longrightarrow P_{b,cl,cv}(L(K,H))$ is Carathéodory.

(H2) There exists a function $\mu\in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,\varphi,\psi)\|_H^2 \leq \mu(t)\left(\|\varphi\|_{\mathcal{B}}^2 + E\|\psi\|_H^2\right), \text{ for any } (\varphi,\psi)\in\mathcal{B}\times H.$$

(H3) There exist constants $d_k>0$, $k=1,2,\dots,m$, such that

$$E\|I_k(x)-I_k(y)\|_H^2 \leq d_k E\|x-y\|_H^2, \text{ for each } x,y\in H,$$

with

$$M^2\sum_{k=1}^m d_k^2 < 1. \tag{3.3}$$

(H4) For each $t\in J$, $a(t,s)$ is measurable on $[0,t]$ and $a(t) = esssup\{|a(t,s)|, 0\leq s\leq t\}$ is bounded on J. The map $t\to a_t$ is continuous from J to $L^\infty(J,\mathbb{R})$, here, $a_t(s) = a(t,s)$.

Set

$$a = \sup_{t\in J} a(t).$$

Theorem 3.6. *Suppose that* (H_φ) *and* $(H1)-(H4)$ *hold. Then the problem (1.1) has at least one mild solution on* J.

Proof. Let $\overline{\phi} : (-\infty, T] \longrightarrow H$ be the extension of ϕ to $(-\infty, T]$ such that $\overline{\phi}(\theta) = \phi(0) = 0$ on J. Consider the space $Y = \{x : (-\infty, T] \to H, x_0 = 0, x|_J \in \mathcal{D}_T\}$ endowed with the uniform convergence topology and define the multi-valued map $\Phi : Y \to \mathcal{P}(Y)$ by $\Phi(h) = \{h \in Y\}$ with

$$h(t) = -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)v(\tau)dw(\tau)ds + \sum_{0<t_k<t} Q(t-t_k)I_k(x(t_k^-)), t \in J.$$

Now we shall show that the operator N satisfies all conditions of Theorem 2.14. For better readability, we break the proof into some steps.

Step 1: Φ has a closed graph.

Let $x_n \to x_*, h_n \in \Phi(x_n)$, and $h_n \to h_*$. We shall show that $h_* \in \Phi(x_*)$. Now, $h_n \in \Phi(x_n)$ means that there exists $v_n \in S_{F,x_{n\rho}}$ such that

$$h_n(t) = -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)v_n(\tau)dw(\tau)ds + \sum_{0<t_k<t} Q(t-t_k)I_k(x_n(t_k^-)),\ t \in J.$$

We must prove that there exists $v_* \in S_{F,x_{*\rho}}$ such that

$$h_*(t) = -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)v_*(\tau)dw(\tau)ds + \sum_{0<t_k<t} Q(t-t_k)I_k(x_*(t_k^-)),\ t \in J.$$

Consider the linear and continuous operator $\Upsilon : L^2(J,H) \longrightarrow C(J,H)$ defined by

$$(\Upsilon v)(t) = \int_0^t \int_0^s R(t-s)a(s,\tau)v(s)dw(\tau)ds.$$

We have

$$\Bigg\| \Bigg(h_n(t) + Q(t)\phi(0) - \sum_{0<t_k<t} Q(t-t_k)I_k(x_n(t_k^-)) \Bigg) - \Bigg(h_*(t) + Q(t)\phi(0) - \sum_{0<t_k<t} Q(t-t_k)I_k(x_*(t_k^-)) \Bigg) \Bigg\| = \|h_n(t) - h_*(t)\| \to 0,\ as\ n \to \infty.$$

From Lemma 2.13 it follows that $\Upsilon \circ S_F$ is a closed graph operator and from the definition of Υ one has

$$h_n(t) + Q(t)\phi(0) - \sum_{0<t_k<t} Q(t-t_k)I_k(x_n(t_k^-)) \in \Upsilon(S_{F,x_{n\rho}}).$$

As $x_n \to x_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,x_{*\rho}}$ such that

$$h_*(t) + Q(t)\phi(0) - \sum_{0<t_k<t} Q(t-t_k)I_k(x_*(t_k^-)) = \int_0^t \int_0^s R(t-s)a(s,\tau)v_*(s)dw(\tau)ds.$$

Hence Φ has a closed graph.

Step 2: We show that the operator Φ is condensing. For this purpose, we decompose Φ as $\Phi_1 + \Phi_2$, where the map $\Phi_1 : \overline{V} \to \mathcal{P}(Y)$ is defined by $\Phi_1(x) = \{h_1 \in Y\}$ with

$$h_1(t) = -Q(t)\phi(0) + \sum_{0<t_k<t} Q(t-t_k)I_k(x(t_k^-)), \quad t \in J,$$

and the map $\Phi_2 : \overline{V} \to \mathcal{P}(Y)$ be defined by $\Phi_2(x) = \{h_2 \in Y\}$ with

$$h_2(t) = \int_0^t \int_0^s R(t-s)a(s,\tau)v(s)dw(\tau)ds, \quad t \in J.$$

First, we show that the operator Φ_1 is a contraction, and second, we prove that Φ_2 is a completely continuous operator. This will be given in several claims.
Claim 1. Φ_1 is a contraction on Y. Let $x, x^* \in Y$ and $h_1 \in \Phi_1(x)$. From (H3), it follows that

$$\begin{aligned}
E\|h_1(t)-h_1^*(t)\|^2 &\leq \sum_{0<t_k<t} E\|Q(t-t_k)I_k(x(t_k^-))-I_k(x^*(t_k^-))\|^2\\
&\leq M^2\sum_{0<t_k<t} E\|I_k(x(t_k^-))-I_k(x^*(t_k^-))\|^2\\
&\leq M^2\sum_{k=1}^m d_k^2 E\|x(t_k^-)-x^*(t_k^-)\|^2\\
&\leq M^2\sum_{k=1}^m d_k^2 E\|x-x^*\|^2.
\end{aligned}$$

Taking the supremum over t,

$$\|h_1-h_1^*\|_Y^2 \leq M^2\sum_{k=1}^m d_k^2\|x-x^*\|_Y^2.$$

By (3.3), the mapping Φ_1 is a contraction.
Claim 2. Φ_2 is convex for each $x \in \overline{V}$.

Indeed, if h_2^1 and h_2^2 belong to Φ_2, then there exist $v_1, v_2 \in S_{F,x_\rho}$ such that, for $t \in J$, we have

$$h_2^i(t) = \int_0^t\int_0^s R(t-s)a(s,\tau)v_i(\tau)dw(\tau)ds, \quad i=1,2.$$

Let $d \in [0,1]$. Then for each $t \in J$, we have

$$dh_2^1(t)+(1-d)h_2^2(t) = \int_0^t\int_0^s R(t-s)a(s,\tau)\left[dv_1(\tau)+(1-d)v_2(\tau)\right]dw(\tau)ds.$$

Since S_{F,x_ρ} is convex (because F has convex values), we have

$$dh_2^1+(1-d)h_2^2 \in \Phi_2.$$

Claim 3. $\Phi_2(\overline{V})$ is completely continuous. We begin by showing $\Phi_2(\overline{V})$ is equicontinuous.
Let $h_2 \in \Phi_2(x)$ for $x \in \overline{V}$ and let $\tau_1, \tau_2 \in [0,T]$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&E\|h_2(\tau_2)-h_2(\tau_1)\|^2\\
&\leq 2E\left\|\int_0^{\tau_1}\int_0^s [R(\tau_2-s)-R(\tau_1-s)]a(s,\tau)v(\tau)dw(\tau)ds\right\|^2\\
&+2E\left\|\int_{\tau_1}^{\tau_2}\int_0^s R(\tau_2-s)a(s,\tau)v(\tau)dw(\tau)ds\right\|^2\\
&\leq 2a^2Tr(Q)\int_0^{\tau_1}\int_0^s\|R(\tau_2-s)-R(\tau_1-s)\|^2\mu(\tau)\left(\|x_\rho(\tau,x_\tau)\|_{\mathcal{B}}^2+E\|x(\tau)\|_H^2\right)d\tau ds\\
&+2a^2Tr(Q)C_{q,M}^2\int_{\tau_1}^{\tau_2}\int_0^s((\tau_2-s))^{2(q-1)}\mu(\tau)\left(\|x_\rho(\tau,x_\tau)\|_{\mathcal{B}}^2+E\|x(\tau)\|_H^2\right)d\tau ds\\
&\leq 2a^2Tr(Q)\int_0^{\tau_1}\int_0^s\|R(\tau_2-s)-R(\tau_1-s)\|^2\mu(\tau)
\end{aligned}$$

$$
\begin{aligned}
&\times \left[2((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}})^2 + 2C_1^* \sup E\|x(\tau)\|_H^2 + E\|x(\tau)\|_H^2\right] d\tau ds \\
&+2a^2 Tr(Q) C_{q,M}^2 \int_{\tau_1}^{\tau_2} \int_0^s (\tau_2 - s)^{2(q-1)} \mu(\tau) \\
&\times \left[2((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}})^2 + 2C_1^* \sup E\|x(\tau)\|_H^2 + E\|x(\tau)\|_H^2\right] d\tau ds \\
&\leq 2a^2 Tr(Q) \left[2(C_2^* + L^\phi)^2\|\phi\|_{\mathcal{B}}^2 + (2C_1^*+1)r^2\right] \int_0^{\tau_1} \int_0^s \|R(\tau_2 - s) - R(\tau_1 - s)\|^2 \mu(\tau) d\tau ds \\
&+2a^2 Tr(Q) C_{q,M}^2 \left[2(C_2^* + L^\phi)^2\|\phi\|_{\mathcal{B}}^2 + (2C_1^*+1)r^2\right] \int_{\tau_1}^{\tau_2} \int_0^s (\tau_2 - s)^{2(q-1)} \mu(\tau) d\tau ds \\
&\leq 2a^2 Tr(Q) \left[2(C_2^* + L^\phi)^2\|\phi\|_{\mathcal{B}}^2 + (2C_1^*+1)r^2\right] \|\mu\|_{L^1} \\
&\times \left[2q^2 \int_0^{\tau_1} \int_0^\infty \sigma \|[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]\xi_q(\sigma) S((\tau_2 - s)^q \sigma)\|^2 d\sigma ds \right.\\
&\left. +2q^2 \int_0^{\tau_1} \int_0^\infty \sigma (\tau_1 - s)^{q-1} \xi_q(\sigma) \|S((\tau_2 - s)^q\sigma) - S((\tau_1 - s)^q \sigma)\|^2 d\sigma ds \right] \\
&+2a^2 Tr(Q) C_{q,M}^2 \left[2(C_2^* + L^\phi)^2\|\phi\|_{\mathcal{B}}^2 + (2C_1^*+1)r^2\right] \int_{\tau_1}^{\tau_2} \int_0^s (\tau_2 - s)^{2(q-1)} \mu(\tau) d\tau ds \\
&\leq 2a^2 Tr(Q) \left[2(C_2^* + L^\phi)^2\|\phi\|_{\mathcal{B}}^2 + (2C_1^*+1)r^2\right] \|\mu\|_{L^1} \\
&\times \left[2C_{q,M}^2 \int_0^{\tau_1} \left|(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\right|^2 ds \right.\\
&\left. +2q^2 \int_0^{\tau_1} \int_0^\infty \sigma (\tau_1 - s)^{q-1} \xi_q(\sigma) \|S((\tau_2 - s)^q\sigma) - S((\tau_1 - s)^q \sigma)\|^2 d\sigma ds \right] \\
&+2a^2 Tr(Q) C_{q,M}^2 \left[2(C_2^* + L^\phi)^2\|\phi\|_{\mathcal{B}}^2 + (2C_1^*+1)r^2\right] \|\mu\|_{L^1} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{2(q-1)} ds.
\end{aligned}
$$

The right hand side tends to zero as $\tau_2 - \tau_1 \to 0$, since $S(t),\ t > 0$ is a strongly continuous semigroup and $S(t)$ is compact for $t > 0$ (so $S(t)$ is continuous in the uniform operator topology for $t > 0$).

Next, we prove that $\Phi_2(\overline{V})(t) = \{h_2(t) : h_2(t) \in \Phi_2(\overline{V})\}$ is relatively compact for every $t \in [0, T]$.

Let $0 < t \leq T$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in \overline{V}$, we define

$$
\begin{aligned}
& h_{2,\varepsilon,\delta}(t) \\
= \quad & q \int_0^{t-\varepsilon} (t-s)^{q-1} \int_\delta^\infty \sigma \xi_q(\sigma) S((t-s)^q \sigma) \int_0^s a(s,\tau) v(\tau) dw(\tau) d\sigma ds \\
= \quad & qS(\varepsilon^q \delta) \int_0^{t-\varepsilon} (t-s)^{q-1} \int_\delta^\infty \sigma \xi_q(\sigma) S((t-s)^q \sigma - \varepsilon^q \delta) \int_0^s a(s,\tau) v(\tau) dw(\tau) d\sigma ds,
\end{aligned}
$$

where $v \in S_{F,x_\rho}$. Since $S(t)$ is a compact operator, the set

$$H_{2,\varepsilon,\delta} = \{h_{2,\varepsilon,\delta}(t) : h_{2,\varepsilon} \in \Phi_2(\overline{V})\}$$

is relatively compact. Moreover, for every $h_2 \in \Phi_2(\overline{V})$ we have

$$
\begin{aligned}
& E\|h_2(t) - h_{2,\varepsilon,\delta}(t)\|^2 \\
& \quad \leq 2qE \left\| \int_0^{t-\varepsilon} (t-s)^{q-1} \int_0^\delta \sigma \xi_q(\sigma) S((t-s)^q \sigma) \int_0^s a(s,\tau) v(\tau) dw(\tau) d\sigma ds \right\|^2
\end{aligned}
$$

$$+2qE\left\|\int_{t-\varepsilon}^{t}(t-s)^{q-1}\int_{0}^{\infty}\sigma\xi_q(\sigma)S((t-s)^q\sigma)\int_0^s a(s,\tau)v(\tau)dw(\tau)d\sigma ds\right\|^2$$
$$\leq 2\frac{(T^qMa)^2}{q}Tr(Q)\left[2(C_2^*+L^\phi)^2\|\phi\|_{\mathcal{B}}^2+(2C_1^*+1)r^2\right]\|\mu\|_{L^1}\int_0^\delta \sigma^2\xi_q^2(\sigma)d\sigma$$
$$+2\left(\frac{\varepsilon^q Ma}{\Gamma(1+q)}\right)^2 Tr(Q)\left[2(C_2^*+L^\phi)^2\|\phi\|_{\mathcal{B}}^2+(2C_1^*+1)r^2\right]\|\mu\|_{L^1}.$$

Hence the set $H_2=\{h_2(t): h_2\in\Phi_2(\overline{V})\}$ is relatively compact. By the Arzelá-Ascoli theorem, we conclude that $\Phi_2(\overline{V})$ is completely continuous.

Step 3: We shall show there exists an open set $V\subseteq Y$, with $x\in\lambda\Phi x$ for $\lambda\in(0,1)$, and $x\notin\partial V$. Let $\lambda\in(0,1)$ and let $x\in\lambda\Phi x$, then there exists an $v\in S_{F,x_\rho}$ such that we have

$$x(t)=-\lambda Q(t)\phi(0)+\lambda\sum_{0<t_k<t}Q(t-t_k)I_k(x(t_k^-))+\lambda\int_0^t\int_0^s R(t-s)a(s,\tau)v(\tau)dw(\tau)ds.$$

Thus, by (H2) and (H3), for each $t\in J$ we have

$$\begin{aligned}
E\|x(t)\|^2 &\leq 3E\|Q(t)\phi(0)\|^2+3\sum_{k=1}^m E\|Q(t-t_k)I_k(x(t_k^-))\|^2\\
&+ 3E\left\|\int_0^t\int_0^s R(t-s)a(s,\tau)v(\tau)dw(\tau)ds\right\|^2\\
&\leq 3M^2E\|\phi(0)\|^2+3M^2\sum_{k=1}^m E\|I_k(x(t_k^-))\|^2\\
&+ 3a^2Tr(Q)C_{q,M}^2\int_0^t\int_0^s(t-s)^{2(q-1)}\mu(\tau)\\
&\times \left[2\left(C_2^*+L^\phi\right)^2\|\phi\|_{\mathcal{B}}^2+(2C_1^*+1)E\|x(\tau)\|^2\right]d\tau ds\\
&\leq 3(MC)^2E\|\phi\|_{\mathcal{B}}^2+3M^2\sum_{k=1}^m E\|I_k(x(t_k^-))-I_k(0)\|^2+3M^2\sum_{k=1}^m E\|I_k(0)\|^2\\
&+ 6a^2Tr(Q)C_{q,M}^2\frac{T^{2q-1}}{2q-1}\|\mu\|_{L^1}\left(C_2^*+L^\phi\right)^2\|\phi\|_{\mathcal{B}}^2\\
&+ 3a^2T(Q)C_{q,M}^2(2C_1^*+1)\|\mu\|_{L^1}\int_0^t(t-s)^{2(q-1)}E\|x(s)\|^2d\tau ds\\
&\leq 3(MC)^2E\|\phi\|_{\mathcal{B}}^2+3M^2\sum_{k=1}^m d_k^2E\|x(t)\|^2+3M^2\sum_{k=1}^m E\|I_k(0)\|^2\\
&+ 6a^2Tr(Q)C_{q,M}^2\frac{T^{2q-1}}{2q-1}\|\mu\|_{L^1}\left(C_2^*+L^\phi\right)^2\|\phi\|_{\mathcal{B}}^2\\
&+ 3a^2Tr(Q)C_{q,M}^2(2C_1^*+1)\|\mu\|_{L^1}\int_0^t(t-s)^{2(q-1)}E\|x(s)\|^2ds.
\end{aligned}$$

Then

$$E\|x(t)\|^2$$

$$
\begin{aligned}
&\leq \frac{3(MC)^2E\|\phi\|_{\mathcal{B}}^2+3M^2\sum_{k=1}^m E\|I_k(0)\|^2+6a^2Tr(Q)C_{q,M}^2\frac{T^{2q-1}}{2q-1}\|\mu\|_{L^1}\left(C_2^*+L^\phi\right)^2\|\phi\|_{\mathcal{B}}^2}{1-3M^2\displaystyle\sum_{k=1}^m d_k^2}\\
&+\frac{3a^2Tr(Q)C_{q,M}^2(2C_1^*+1)\|\mu\|_{L^1}}{1-3M^2\displaystyle\sum_{k=1}^m d_k^2}\int_0^t(t-s)^{2(q-1)}E\|x(s)\|^2ds\\
&\leq \theta_1+\theta_2\int_0^t(t-s)^{2(q-1)}E\|x(s)\|^2ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1 &= \frac{3(MC)^2E\|\phi\|_{\mathcal{B}}^2+3M^2\sum_{k=1}^m E\|I_k(0)\|^2+6a^2Tr(Q)C_{q,M}^2\frac{T^{2q-1}}{2q-1}\|\mu\|_{L^1}\left(C_2^*+L^\phi\right)^2\|\phi\|_{\mathcal{B}}^2}{1-3M^2\displaystyle\sum_{k=1}^m d_k^2}\\
\theta_2 &= \frac{3a^2Tr(Q)C_{q,M}^2(2C_1^*+1)\|\mu\|_{L^1}}{1-3M^2\displaystyle\sum_{k=1}^m d_k^2}.
\end{aligned}
$$

Therefore, in view of Lemma 2.12, we have for all $t\in J$,

$$
\begin{aligned}
E\|x(t)\|^2 &\leq \theta_1\left[1+\int_0^t\sum_{n=1}^\infty\frac{(\theta_2\Gamma(2q-1))^n}{\Gamma(n(2q-1))}(t-s)^{n(2q-1)-1}ds\right]\\
&\leq \theta_1\left[1+\sum_{n=1}^\infty\frac{(\theta_2\Gamma(2q-1))^n}{n(2q-1)\Gamma(n(2q-1))}T^{n(2q-1)}\right]\\
&\leq \theta_1\left[1+\sum_{n=1}^\infty\frac{\left(\theta_2\Gamma(2q-1)T^{(2q-1)}\right)^n}{\Gamma(n(2q-1)+1)}\right]\\
&\leq \theta_1E_{2q-1}\left(\theta_2\Gamma(2q-1)T^{(2q-1)}\right)^n,
\end{aligned}
$$

where $E_{2q-1}\left(\theta_2\Gamma(2q-1)T^{(2q-1)}\right)^n=\displaystyle\sum_{n=0}^\infty\frac{\left(\theta_2\Gamma(2q-1)T^{(2q-1)}\right)^n}{\Gamma(n(2q-1)+1)}$ is the Mittag-Leffer function. This implies that

$$\|x(t)\|_Y^2\leq\theta_1E_{2q-1}\left(\theta_2\Gamma(2q-1)T^{(2q-1)}\right)^n.$$

Then there exists M^* such that $\|x(t)\|_Y^2\neq M^*$. Set

$$V=\{z\in Y:\|x(t)\|_Y^2<M^*\}.$$

From the choice of V, there is no $x\in\partial V$ such that $x\in\lambda\Phi x$ for $\lambda\in(0,1)$. As a consequence of Theorem 2.14, we deduce that Φ has a fixed point x defined on $(-\infty,T]$, which is mild solution of problem (1.1). This completes the proof. □

4. An Example

We consider the impulsive fractional partial stochastic functional integro-differential inclusions of the form:

$$\begin{aligned}\frac{\partial^q}{\partial t^q}v(t,\zeta) &\in \frac{\partial^2}{\partial \zeta^2}v(t,\zeta) + \int_{-\infty}^{t} a_1(s-t)v(s-\rho_1(t)\rho_2(|v(t)|),\xi)dw(s)\\ &\quad + \int_0^t (t-s)^2\cos|v(s,\zeta)|ds\\ v(t,0) &= v(t,\pi) = 0\\ v(\theta,\zeta) &= v_0(\theta,\zeta), \qquad -\infty<\theta\le 0,\\ \Delta v(t_k)(\zeta) &= \int_{-\infty}^{t_k} p_k(t_k-y)dy\cos(v(t_k)(\zeta))\end{aligned} \tag{4.1}$$

where $0<q<1, t\in[0,T], \zeta\in[0,\pi], v_0: \ (-\infty,0]\times[0,\pi]\to\mathbb{R},\ p_k:\mathbb{R}\to\mathbb{R}, k=1,2,\dots,m$, and the functions $a_1:\mathbb{R}\to\mathbb{R}, \rho_i:[0,+\infty)\to[0,+\infty),\ i=1,2$ are continuous functions.

Set $H=L^2([0,\pi]), A:D(A)\subset X\to X$ is the map defined by $A\omega=\omega''$ with domain

$$D(A)=\{\omega\in H:\omega,\omega' \text{ are absolutely continuous, } \omega''\in H, \omega(0)=\omega(\pi)=0\},$$

then

$$A\omega=\sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \quad \omega\in D(A),$$

where $\omega_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx), n\in\mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\ge0}$ in H and is given by

$$S(t)\omega=\sum_{n=1}^{\infty} e^{-n^2t}(\omega,\omega_n)\omega_n, \quad \forall\omega\in E,\ \textit{and every } t>0.$$

From these expressions, it follows that $\{S(t)\}_{t\ge0}$ is a uniformly bounded compact semigroup.

For the phase space, we choose $\mathcal{B}=C_0\times L^2(g,X)$, see Example 2.8 for details.

Set

$$x(t)(\zeta)=v(t,\zeta), \qquad t\in[0,T], \zeta\in[0,\pi].$$

$$\phi(\theta)(\zeta)=v_0(\theta,\zeta), \qquad \theta\in(-\infty,0], \zeta\in[0,\pi].$$

$$a(t,s)=(t-s)^2,$$

$$f(t,\varphi,x(t))(\zeta)=\int_{-\infty}^{0} a_1(s)\varphi(s,\xi)ds+\cos|x(t)(\zeta)|, \quad t\in[0,T], \zeta\in[0,\pi].$$

$$\rho(t,\varphi)=s-\rho_1(s)\rho_2(\|\varphi(0)\|).$$

$$I_k(x(t_k^-))(\zeta)=\int_{-\infty}^{0} p_k(t_k-y)dy\cos(x(t_k)(\zeta)), \quad k=1,2,\dots,m.$$

Thus, problem (4.1) can be rewritten as the abstract problem (1.1). The following result is a direct consequence of Theorem 3.6.

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and let $t \to \varphi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then there exists a mild solution of (4.1).*

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[2] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[3] R.P. Agarwal, M. Belmekki and M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Adv. Difference Equ.* **2009** (2009), Article ID 981728, 47 pages.

[4] R.P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.* **109** (2010), 973-1033.

[5] R.P. Agarwal, M. Benchohra and B.A. Slimani, Existence results for differential equations with fractional order impulses, *Mem. Differential Equations Math. Phys.* **44** (2008), 1-21.

[6] D.D. Bainov and P.S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[7] D. Baleanu, J.A.T. Machado and A.C.J. Luo, *Fractional Dynamics and Control*, Springer, New York, 2012.

[8] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol. 2, New York, 2006.

[9] M. Benchohra and B.A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, *Electron. J. Differential Equations* **2009** (10) (2009), 1-11.

[10] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, (1992).

[11] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications (Second Edition)*, CRC Press, 2007.

[12] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[13] K. Diethelm, *The Analysis of Fractional Differential Equations*. Springer, Berlin, 2010.

[14] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[15] W. Grecksch and C. Tudor, *Stochastic Evolution Equations: a Hilbert Space Approach*, Akademic-Verlag, Berlin, 1995.

[16] T. Guendouzi and O. Benzatout, Existence of solutions for fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators, *Viet. J. Math.* **43** (2015), 687-704.

[17] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funk. Ekvacioj*, **21** (1978), 11-41.

[18] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture notes in Math. Vol 840, Springer-Verlag, New York/Berlin, 1981.

[19] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlinear Anal. RWA* **7** (2006), 510-519.

[20] R. Hilfer, *Applications of Fractional Calculus in Physics*. Singapore, World Scientific, 2000.

[21] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.

[22] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Volume I: Theory, Kluwer Academic Publishers, Dordrecht, Boston, London, 1997.

[23] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam, 2006.

[24] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, NJ, 1989.

[25] V. Lakshmikantham, S. Leela and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[26] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781-786.

[27] A. Lin and L. Hu, Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions, *Comput. Math. Appl* **59** (2010), 64-73.
[28] F. Mainardi, P. Paradisi and R. Gorenflo, *Probability distributions generated by fractional diffusion equations*, in Econophysics: An Emerging Science, J. Kertesz and I. Kondor, Eds., Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
[29] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Limited, Chichester, 1997.
[30] D. O'Regan, Nonlinear alternative for multivalued maps with applications to operator inclusions in abstract spaces. *Proc. Amer. Math. Soc.* **127** (1999), 3557-3564.
[31] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1993.
[32] Y.V. Rogovchenko, Nonlinear impulsive evolution systems and applications to population models, *J. Math. Anal. Appl.* **207** (1997), 300-315.
[33] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[34] A.M. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations, World Scientific*, Singapore, 1995.
[35] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
[36] Z. Yan and H. Zhang, Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay, *Electron. J. Differential Equations* **2013** (81) (2013), 1-21.
[37] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **338** (2007) 1075-1081.

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