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NEW NUMERICAL APPROACH FOR SOLVING FRACTIONAL DIFFERENTIAL- ALGEBRAIC EQUATIONS

HODA F. AHMED, MARINA B. MELAD

Abstract. This paper introduces a new approach for solving fractional differential algebraic equations (FDAEs) by using the operational matrix of Rieman Liouville (RL) fractional integral of the shifted Gegenbauer polynomials. By using the new shifted Gegenbauer operational matrix (SGOM) of RL fractional integral, the FDAEs are transformed into a system of algebraic equations which are easily to solve. Numerical examples associated by numerical comparisons with other methods in the literature are introduced to illustrate the efficiency and accuracy of the proposed approach.

1. Introduction

Recently, fractional calculus (FC) has made a scientific revolution in the traditional calculus. This is due to its several applications in many different scientific fields like physics, chemistry, engineering, and etc. These applications are expressed in the form of fractional differential equations (FDEs) or fractional differential- algebraic equations (FDAEs) [1]-[4]. Many numerical methods are investigated to present accurate numerical solutions for such problems, since most of these problems don't have exact solutions. The methods such as Adomian decomposition method [5]-[7], variational iteration method [8]-[10], spectral methods [11, 12] are widely used in solving FDEs and FDAEs. Many physical applications are obviously designated by systems of DAEs. These types of systems follow in the modeling of power systems, electrical networks, optimal control, mechanical systems subject to constraints, chemical process and in other numerous applications. Various numerical approaches for approximating the solutions of DAEs have been presented in [13]-[18]. Many important mathematical models can be expressed in terms of FDAEs. So various numerical techniques are developed to solve these problems. In this respect we refer to [19]-[23]. For more decades, spectral methods have obtained a great interest in solving differential equations. These methods are characterized by their precision for any number of unknowns. There are three main spectral images, they are the Galerkin, collocation and Tau methods [24]. In the spectral methods, the explicit formula for operational matrices of fractional integrals and

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derivatives for classical orthogonal polynomials are needed. Orthogonal functions have attacked significant importance in dealing with various problems of differential equations (DEs). By using these functions, the DEs are transformed into systems of algebraic equations. Some types of orthogonal polynomials have been introduced as basis functions of the operational matrices of fractional derivatives and integrals which are used to solve ordinary and partial fractional differential equations [25]-[32]. Ultraspherical (Gegenbauer) polynomials have many useful properties. They achieve rapid rates of convergence for small range of the spectral expansion terms [33]-[37]. This encourages many authors for applying these polynomials for solving different kinds of DEs and FDEs. In this respect, we refer to [38]-[41] and [42, 43] respectively.

In the present paper we investigate the operational matrix of the RL fractional integral of the shifted Gegenbauer polynomials and use it with the Tau method to present a numerical solution to the following FDAEs

$$D^{\nu_i} y_i(t) = f(t, y_1, ..., y_n, y_1', ..., y_n'), i = 1, 2, ..., m-1, t \geq 0, 0 \leq \nu_i \leq 1, \quad (1)$$

$$y_i(t) = g_i(t, y_1, ..., y_n), i = m, m+1, ..., n, \quad (2)$$

with the initial conditions

$$y_i(0) = d_i, i = 1, ..., n, \quad (3)$$

Where D^ν is the RL fractional derivative. The analysis of the existence and uniqueness of the FDEs and FDAEs have been introduced in [44]-[46] and [47], respectively.

The paper is organized as follows. In section 2 we review some necessary definitions and properties of fractional calculus and ultraspherical (Gegenbauer) polynomials. In section 3 the SGOM of fractional integration is derived. In section 4 the convergence of the proposed method is discussed. In section 5 the proposed mechanism of applying SGOM of fractional integration for solving FDAEs is discussed. In section 6 the proposed method is used to solve several problems of FDAEs. Finally conclusions are given in section 7.

2. Preliminaries and Definitions

2.1 Fractional Calculus Definitions

Definition 1 One of the popular definitions of fractionl integral is the RL, which is defined by

$$\begin{aligned} &I^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-\xi)^{\nu-1} f(\xi) d\xi, m-1 < \nu < m, m \in N, \nu > 0, \, x > 0, \\ &I^0 f(x) = f(x). \end{aligned} \quad (4)$$

For more properties about RL fractional integral, see [48], we just recall the next property

$$I^\nu t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\nu+\beta+1)} t^{\nu+\beta}. \quad (5)$$

Definition 2 D^ν is the RL fractional derivative of order ν which defined by

$$D^\nu f(t) = \frac{d^m}{dt^m}(I^{m-\nu} f(t)), m-1 < \nu \leq m, m \in N, \nu \in R, \quad (6)$$

where m is the smallest integer order greater than ν.

Lemma 1 If $m-1 < \nu \leq m$, $m \in N$, then

$$(D^\nu I^\nu) f(t) = f(t),$$

$$(I^{\nu}D^{\nu})\,f(t)=f(t)-\sum_{i=0}^{m-1}f^{(i)}(0^{+})\frac{t^{i}}{i!},t>0. \tag{7}$$

2.2 Shifted ultraspherical (Gegenbauer) polynomials and their properties

The ultraspherical (Gegenbauer) polynomials $C_j^{(\alpha)}(t)$, of degree $j\in\mathbb{Z}^{+}$, and associated with the parameter ($\alpha>\frac{-1}{2}$) are a sequence of real polynomials in the finite domain $[-1,1]$. They are a family of orthogonal polynomials which has many applications.

Definition 1 The ultraspherical (Gegenbauer) polynomials are a special class of Jacobi polynomials $P_j^{(\alpha,\beta)}$, with $\alpha=\beta=\alpha-\frac{1}{2}$ so that

$$C_j^{(\alpha)}(t)=\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(j+2\alpha)}{\Gamma(2\alpha)\Gamma(j+\alpha+\frac{1}{2})}P_j^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(t),j=0,1,2,........$$

- There are useful relations to the Chebyshev polynomials of the first kind $T_j(t)$, second kind $U_j(t)$ and the Legender polynomials $L_j(t)$ with the Gegenbaure polynomials as follows

$$T_j(t)\equiv\frac{j}{2}\lim_{\alpha\to 0}\alpha^{-1}C_j^{(\alpha)}(t),j\geq 1,$$

$$U_j(t)\equiv(j+1)C_j^{(1)}(t),$$

and

$$L_j(t)\equiv C_j^{(\frac{1}{2})}(t),$$

respectively.

- Ultraspherical polynomials are eigenfunctions of the following singular Sturm-Liouville equation

$$(1-t^2)\frac{d^2}{dt^2}\phi_j(t)-(2\alpha+1)t\frac{d}{dt}\phi_j(t)+j(j+2\alpha)\phi_j(t)=0,$$

and may be generated using the recurrence equation

$$(j+2\alpha)C_{j+1}^{(\alpha)}(t)=2(j+\alpha)tC_j^{(\alpha)}(t)-jC_{j-1}^{(\alpha)}(t),j=1,2,..........$$

with

$$C_0^{(\alpha)}(t)=1,C_1^{(\alpha)}(t)=t.$$

- The ultraspherical polynomials can be obtained from the Rodrigues$'$ formula

$$C_j^{(\alpha)}(t)=(\frac{-1}{2})^{j}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(j+\alpha+\frac{1}{2})}(1-t^2)^{\frac{1}{2}-\alpha}\frac{d^j}{dt^j}\left[(1-t^2)^{j+\alpha-\frac{1}{2}}\right].$$

- The orthogonality relation of the Gegenbauer polynomials is given by the weighted inner product

$$\left\langle C_i^{(\alpha)}(t),C_j^{(\alpha)}(t)\right\rangle=\int_{-1}^{1}C_i^{(\alpha)}(t)C_j^{(\alpha)}(t)\omega^{(\alpha)}(t)dt=\lambda_j^{(\alpha)}\delta_{i,j},$$

where $\omega^{(\alpha)}(t)$ is the weight function, it is an even function given from relation

$$\omega^{(\alpha)}(t)=(1-t^2)^{\alpha-\frac{1}{2}},$$

and

$$\lambda_j^{(\alpha)} = \|C_j^{(\alpha)}(t)\|^2 = \frac{2^{1-2\alpha}\pi\Gamma(j+2\alpha)}{j!(j+\alpha)\Gamma^2(\alpha)},$$

is the normalization factor and $\delta_{i,j}$ is the Kronecker delta function.

- To use these polynomials in the interval $[0, L]$, the shifted Gegenbauer polynomials are formed by replacing the variable t with $\frac{2t}{L} - 1, 0 \leq t \leq L$. All results of ultraspherical polynomials can be easily transformed to give the corresponding results for their shifted ones. The shifted Gegenbauer can be written as

$$C_{S,j}^{(\alpha)}(t) = C_j^{(\alpha)}(\frac{2t}{L} - 1), C_{S,0}^{(\alpha)}(t) = 1, C_{S,1}^{(\alpha)}(t) = \frac{2t}{L} - 1.$$

- It's analytical form is given by

$$\begin{aligned} C_{S,j}^{(\alpha)}(t) &= \sum_{k=0}^{j} (-1)^{j-k} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(j+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(j-k)!k!L^k} t^k, \\ C_{S,j}^{(\alpha)}(0) &= (-1)^j \frac{\Gamma(j+2\alpha)}{\Gamma(2\alpha)j!}. \end{aligned} \tag{8}$$

- The orthogonal relation of shifted Gegenbauer polynomials is getting from

$$\left\langle C_{S,i}^{(\alpha)}(t), C_{S,j}^{(\alpha)}(t) \right\rangle = \int_0^L C_{S,i}^{(\alpha)}(t) C_{S,j}^{(\alpha)}(t) \omega_S^{(\alpha)}(t) dt = \lambda_{S,j}^{(\alpha)} \delta_{i,j}, \tag{9}$$

where $\omega_S^{(\alpha)}(t)$ is the weight function, it is an even function given from the relation

$$\omega_S^{(\alpha)}(t) = (tL - t^2)^{\alpha - \frac{1}{2}},$$

and

$$\lambda_{S,j}^{(\alpha)} = \left(\frac{L}{2}\right)^{2\alpha} \lambda_j^{(\alpha)}.$$

- These polynomials recover the shifted Chebyshev polynomials of the first kind $T_{S,j}(t) \equiv C_{S,j}^{(0)}(t)$, the shifted Legendre polynomials $L_{S,j}(t) \equiv C_{S,j}^{(\frac{1}{2})}(t)$, and the shifted Chebyshev polynomials of the second kind $C_{S,j}^{(1)}(t) \equiv \frac{1}{j+1} U_{S,j}(t)$.

- The q times repeated derivative of the shifted Gegenbauer polynomials given from the relation

$$D^q C_{S,j}^{(\alpha)}(t) = \frac{2^{2q}(\alpha+q-1)!}{(\alpha-1)!} C_{S,j-q}^{(\alpha+q)}(t), \tag{10}$$

by substituting $t = 0$ at (10), we get a bout the relation

$$D^q C_{S,j}^{(\alpha)}(0) = \frac{(-1)^{j-q} 2^{2q} (\alpha+q-1)! \Gamma(j+q+2\alpha)}{(\alpha-1)! \Gamma(2\alpha+2q)(j-q)!}. \tag{11}$$

- The square integrable function $y(t)$ in $[0, L]$ can be approximated by shifted Gegenbauer polynomials as:

$$y_N(t) = \sum_{j=0}^{N} \tilde{y}_j C_{S,N,j}^{(\alpha)}(t), \tag{12}$$

where the coefficients $\tilde{y}_j$ are computed from

$$\tilde{y}_j = (\lambda_{S,j}^{(\alpha)})^{-1} \int_0^L y(t)\omega_S^{(\alpha)}(t) C_{S,N,j}^{(\alpha)}(t)dt. \tag{13}$$

- The approximation of function $y(t)$ can be written in the vector form as

$$y_N(t) = Y^T \phi(t), \tag{14}$$

where $Y^T = [\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_N]$ is the shifted Gegenbauer coefficient vector, and

$$\phi(t) = \left[C_{S,N,0}^{(\alpha)}(t), C_{S,N,1}^{(\alpha)}(t), ..., C_{S,N,N}^{(\alpha)}(t) \right]^T \tag{15}$$

is the shifted Gegenbauer vector.

- The q times repeated integration of the Gegenbauer vector is computed from

$$I^q \phi(t) \simeq P^q \phi(t), \tag{16}$$

where P^q is called the operational matrix (OM) of the integration of order q.

3. Operational Matrix of Fractional Integration of the Shifted Gegenbauer Polynomials

In this section, shifted Gegenbauer operational matrix (SGOM) of RL fractional integral will be proved.

Theorem 1 Let $\phi(t)$ be the shifted Gegenbauer vector and $\nu > 0$ then

$$I^\nu \phi(t) \simeq P^{(\nu)} \phi(t), \tag{17}$$

where $t \in [0, L]$ and $P^{(\nu)}$ is called OM of fractional integration of order ν in the RL sense, it is a square matrix of order $(N+1) \times (N+1)$ and is written as follows:

$$P^{(\nu)} = \begin{pmatrix} \sum_{k=0}^{0} \xi_{0,0,k} & \sum_{k=0}^{0} \xi_{0,1,k} & \cdot\ \cdot\ \cdot & \sum_{k=0}^{0} \xi_{0,N,k} \\ \sum_{k=0}^{1} \xi_{1,0,k} & \sum_{k=0}^{1} \xi_{1,1,k} & \cdot\ \cdot\ \cdot & \sum_{k=0}^{1} \xi_{1,N,k} \\ \vdots & \vdots & \cdot\ \cdot\ \cdot & \vdots \\ \sum_{k=0}^{i} \xi_{i,0,k} & \sum_{k=0}^{i} \xi_{i,1,k} & \cdot\ \cdot\ \cdot & \sum_{k=0}^{i} \xi_{i,N,k} \\ \vdots & \vdots & \cdot\ \cdot\ \cdot & \vdots \\ \sum_{k=0}^{N} \xi_{N,0,k} & \sum_{k=0}^{N} \xi_{N,1,k} & \cdot\ \cdot\ \cdot & \sum_{k=0}^{N} \xi_{N,N,k} \end{pmatrix} \tag{18}$$

where $\xi_{i,j,k}$ is given by:

$$\xi_{i,j,k} = \Xi \times \Upsilon,$$

where

$$\Xi = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2})\Gamma(2\alpha)\Gamma(k + \nu + 1)(i - k)! L^k},$$

$$\Upsilon = \sum_{f=0}^{j} (-1)^{j-f} \frac{j!(j + \alpha)\Gamma^2(\alpha)\Gamma^2(\alpha + \frac{1}{2})\Gamma(2\alpha + j + f)\Gamma(\nu + k + f + \alpha + \frac{1}{2})}{2^{(1-4\alpha)}\pi\Gamma(2\alpha + j)\Gamma(2\alpha)\Gamma(\alpha + f + \frac{1}{2})(j - f)! f! \Gamma(\nu + k + f + 2\alpha + 1) L^k} \tag{19}$$

Proof. From relation (8) and by using Eqs. (4) and (5), we can write

$$I^{\nu}C_{S,i}^{(\alpha)}(t)=\sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!k!L^{k}}I^{\nu}(t^{k}),t\in[0,L]$$

$$=\sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!\Gamma(\nu+k+1)L^{k}}t^{k+\nu},i=0,1,2,....,N. \tag{20}$$

The function $t^{k+\nu}$ can be written as a series of $N+1$ terms of Gegenbauer polynomial,

$$t^{k+\nu}=\sum_{j=0}^{N}\tilde{t}_{j}C_{S,j}^{(\alpha)}(t), \tag{21}$$

Where

$$\tilde{t}_{j}=\sum_{f=0}^{j}(-1)^{j-f}\frac{j!(j+\alpha)\Gamma^{2}(\alpha)\Gamma^{2}(\alpha+\frac{1}{2})\Gamma(2\alpha+j+f)\Gamma(\nu+k+f+\alpha+\frac{1}{2})}{2^{(1-4\alpha)}\pi\Gamma(2\alpha+j)\Gamma(\alpha+f+\frac{1}{2})(j-f)!f!\Gamma(2\alpha)\Gamma(\nu+k+f+2\alpha+1)L^{k}}. \tag{22}$$

Now, by employing equations (20)-(22) we obtain

$$I^{\nu}C_{S,i}^{(\alpha)}(t)=\sum_{k=0}^{i}\sum_{j=0}^{N}(-1)^{i-k}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!\Gamma(\nu+k+1)L^{k}}\tilde{t}_{j}C_{S,j}^{(\alpha)}(t),$$

$$=\sum_{j=0}^{N}\left(\sum_{k=0}^{i}\xi_{i,j,k}\right)C_{S,j}^{(\alpha)}(t),i=0,1,....,N, \tag{23}$$

where $\xi_{i,j,k}$ is given in Eq. (19). Writing the last equation in a vector form gives

$$I^{\nu}C_{S,i}^{(\alpha)}(t)\simeq\left[\sum_{k=0}^{i}\xi_{i,0,k},\sum_{k=0}^{i}\xi_{i,1,k},...,\sum_{k=0}^{i}\xi_{i,N,k}\right]\phi(t),i=0,1,...,N, \tag{24}$$

which finishes our proof.

4. Error and Convergence Analysis

4.1 Error Bound

Theorem 2 Suppose that $H=L^{2}[0,1]$ is the Hilbert space, and let Y be a closed subspace of H such that $Y=Span\left\{C_{S,0}^{(\alpha)}(t),C_{S,1}^{(\alpha)}(t),...,C_{S,N}^{(\alpha)}(t)\right\}$. Let $f(t)\in C^{n+1}[0,1]$, if $\sum_{j=0}^{N}\tilde{t}_{j}C_{S,j}^{\alpha}(t)$ is the best approximation of $f(t)$ out of Y then:

$$\parallel f(t)-\sum_{j=0}^{N}\tilde{t}_{j}C_{S,j}^{\alpha}(t)\parallel\leq\frac{h^{\frac{2n+3}{2}}R}{(n+1)!\sqrt{2n+3}},t\in[t_{i},t_{i+1}]\subseteq[0,1],$$

where $R=\max_{t\in[t_{i},t_{i+1}]}|f^{(n+1)}(t)|$ and $h=t_{i+1}-t_{i}$.

Proof. We set

$$y_{1}(t)=f(t_{i})+f^{'}(t_{i})(t-t_{i})+f^{''}(t_{i})\frac{(t-t_{i})^{2}}{2!}+...+f^{(n)}(t_{i})\frac{(t-t_{i})^{n}}{n!}.$$

From Taylor's expansion it is clear that

$$|f(t)-y_{1}(t)|\leq|f^{(n+1)}(\xi_{t})|\frac{(t-t_{i})^{n+1}}{(n+1)!},$$

where $\xi_t \in [t_i, t_{i+1}]$. Since $\sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t)$ is the best approximation of $f(t)$ out of Y and $y_1(t) \in Y$, then we have

$$\| f(t) - \sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t) \|_2^2 \leq \| f(t) - y_1(t) \|_2^2 = \int_{t_i}^{t_{i+1}} |f(u) - y_1(u)|^2 du,$$

$$\leq \int_{t_i}^{t_{i+1}} \| f^{(n+1)}(\xi_t) \|^2 \frac{(u - t_i)^{(n+1)}}{(n+1)!} du,$$

$$\leq \frac{h^{2n+3} R^2}{((n+1)!)^2 (2n+3)}.$$

Taking the square root of both sides, we have

$$\| f(t) - \sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t) \|_2 \leq \frac{h^{\frac{2n+3}{2}} R}{((n+1)!) \sqrt{(2n+3)}}.$$

which is the desired result. Hence we conclude that at each subinterval $[t_i, t_{i+1}], i = 1, 2, ..., n$. $f(t)$ has a local error bound of $O(h^{\frac{2n+3}{2}})$. Thus, $f(t)$ has a global error of $O(h^{\frac{2n+1}{2}})$ on the whole interval $[0, 1]$.

In the following theorem, the error estimate for the approximated functions will be expressed in terms of Gram determinant [49].

Theorem 3 Let $y(t)$ be an arbitrary element of H and $y^*(t)$ be the unique best approximation of $y(t)$ out of Y, then

$$\| y(t) - y^*(t) \|^2 = \frac{Gram(y(t), C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))}{Gram(C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))} \tag{25}$$

where
$Gram(y(t), C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))$

$$= \begin{vmatrix} < y(t), y(t) > & < y(t), C^{(\alpha)}_{S,0}(t) > & . \quad . \quad . & < y(t), C^{(\alpha)}_{S,N}(t) > \\ < C^{(\alpha)}_{S,0}(t), y(t) > & < C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,0}(t) > & . \quad . \quad . & < C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,N}(t) > \\ < C^{(\alpha)}_{S,1}(t), y(t) > & < C^{(\alpha)}_{S,1}(t), C^{(\alpha)}_{S,0}(t) > & . \quad . \quad . & < C^{(\alpha)}_{S,1}(t), C^{(\alpha)}_{S,N}(t) > \\ . & . & & . \\ . & . & . \quad . \quad . & . \\ . & . & & . \\ < C^{(\alpha)}_{S,N}(t), y(t) > & < C^{(\alpha)}_{S,N}(t), C^{(\alpha)}_{S,0}(t) > & . \quad . \quad . & < C^{(\alpha)}_{S,N}(t), C^{(\alpha)}_{S,N}(t) > \end{vmatrix}$$

4.2 Convergence Analysis

Consider the error, E_{I^ν} of the operational matrix of integration in the RL sense as

$$E_{I^\nu} = P^\nu \Phi(t) - I^\nu \Phi(t),$$

where

$$E_{I^\nu} = [E_{I^\nu,0}, E_{I^\nu,1}, ., ., ., E_{I^\nu,N}]^T,$$

is an error vector. From Eq. (20), we had approximated $t^{k+\nu}$ as $\sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t)$. From theorem 3 we have

$$\left\| t^{k+\nu} - \sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t) \right\|_2 = \left(\frac{Gram(t^{k+\nu}, C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))}{Gram(C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))} \right)^{\frac{1}{2}} \tag{26}$$

From Eq. (23), we obtain the upper bound of the operational matrix of integration as follows

$$\|E_{I^{\nu},i}\|_2 = \left\| I^{\nu} C^{\alpha}_{S,i}(t) - \sum_{j=0}^{N} \left(\sum_{k=0}^{i} \xi_{i,j,k} \right) C^{(\alpha)}_{S,j}(t) \right\|, i = 0, ..., N, \tag{27}$$

$$\leq \sum_{k=0}^{i} \left| \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2})\Gamma(2\alpha)(i - k)!\Gamma(\nu + k + 1)} \right| \left(\frac{Gram(t^{k+\nu}, C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))}{Gram(C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), ..., C^{(\alpha)}_{S,N}(t))} \right)^{\frac{1}{2}} \tag{28}$$

The following theorem illustrate that with increasing the number of GPs, the error tends to zero.

Theorem 4 Suppose that function $y(t) \in L^2[0,1]$ is approximated by $g_N(t)$ as follows

$$g_N(t) = \mu_0 C^{\alpha}_{S,0}(t) + \mu_1 C^{\alpha}_{S,1}(t) + ... + \mu_N C^{\alpha}_{S,N}(t),$$

where

$$\mu_i = \int_0^1 C^{\alpha}_{S,i}(t) y(t) dt, i = 0, ..., N.$$

Consider

$$s_N(y) = \int_0^1 [y(t) - g_N(t)]^2 \, dt,$$

then we have

$$\lim_{N\to\infty} s_N(y) = 0.$$

For the proof see [50].

5. SGOM of Fractional Integration for Solving Fractional Differential Algebraic Equations

In this section, we use SGOM of integration to solve FDAEs (1) and (2) with the initial condition (3). Firstly, we apply the RL integral of order ν_i on Eqs.(1) and by using Eq.(7), we get

$$y_i(t) - d_i = f(I^{\nu_i} t, I^{\nu_i} y_1, ..., I^{\nu_i} y_n, I^{\nu_i - 1}[y_1 - d_1], ..., I^{\nu_i - 1}[y_n - d_n]), \tag{29}$$

Secondly, we approximate $y_i(t)$, $I^{\nu_i} y_j(t)$ $(j = 1, ..., n)$ in (29) and (2) by using shifted Gegenbauer polynomials in (14) and (17), we get

$$Y_i^T \phi(t) - d_i = f(I^{\nu_i} t, Y_1^T P^{\nu_i} \phi(t), ..., Y_n^T P^{\nu_i} \phi(t), Y_1^T P^{\nu_i - 1} \phi(t) - d_1, ..., Y_n^T P^{\nu_i - 1} \phi(t) - d_n), \tag{30}$$

$$Y_i^T \phi(t) = g_i(t, Y_1^T \phi(t), ..., Y_n^T \phi(t)), \tag{31}$$

where $I^{\nu_i} t$ can be calculated from relation (4). The initial condition is approximated as

$$y_i(0) = Y_i^T P^{\nu_i} \phi(0) = d_i, i = 1, 2, ..., n. \tag{32}$$

So, the residuals for the system (29) and (2) can be written as

$$R1_{N,i}(t) = \left(Y_i^T - f(Y_1^T P^{\nu_i}, ..., Y_n^T P^{\nu_i}, Y_1^T P^{\nu_i - 1}, ..., Y_n^T P^{\nu_i - 1}) - Q_i^T\right) \phi(t), \quad (33)$$

$$R2_{N,i}(t) = \left(Y_i^T - g_i(Y_1^T, ..., Y_n^T)\right) \phi(t), \quad (34)$$

where $Q_i^T = [q_{i0}, q_{i1}, ..., q_{iN}]$ and $\Psi_i^T = [\psi_{i0}, \psi_{i1}, ..., \psi_{iN}]$ are known vectors given from the relation (13), as $q_i = d_i - I^{\nu_i} t + \sum_{j=1}^{n} d_j$. By using Tau method, we generate N algebraic equations as

$$< R1_{N,i}(t), C^{(\alpha)}_{S,N,j} >= \int_0^1 R1_{N,i}(t) C^{(\alpha)}_{S,N,j}(t) dt = 0, i = 1, ..., m-1, j = 0, 1, ..., N-1. \quad (35)$$

$$< R2_{N,i}(t), C^{(\alpha)}_{S,N,j} >= \int_0^1 R2_{N,i}(t) C^{(\alpha)}_{S,N,j}(t) dt = 0, i = m, m+1, ..., n, j = 0, 1, ..., N-1. \quad (36)$$

From Eqs.(35)-(36) and Eq. (32), $n(N+1)$ set of algebraic equations are generated. This algebraic system can be solved easily. Consequently the approximate solution $y_i(t)$ can be obtained.

6. Illustrative Problems

In this section, some problems are given to illustrate the applicability and accuracy of the proposed mechanism.

Problem 1 Consider the following linear FDAEs [21]

$$D^{\nu} y_1(t) + y_1(t) - y_2(t) = -\sin(t), 0 < \nu \leq 1, \quad (37)$$

$$y_1(t) + y_2(t) = e^{-t} + \sin(t), \quad (38)$$

with initial conditions

$$\begin{aligned} y_1(0) &= 1, \\ y_2(0) &= 0, \end{aligned} \quad (39)$$

the exact solution is $y_1(t) = e^{-t}$, $y_2(t) = \sin(t)$ when $\nu = 1$.

The problem can be simplified by substituting Eq. (38) into Eq. (37) to get

$$D^{\nu} y_1(t) + 2y_1(t) = e^{-t}, \quad (40)$$

From Eq. (12) the approximate solution with N=9, is written as

$$y_1(t) = \sum_{j=0}^{9} \tilde{y}_{1,j} C^{(\alpha)}_{S,N,j}(t), \quad (41)$$

and

$$e^{-t} = \sum_{j=0}^{9} \tilde{g}_{1,j} C^{(\alpha)}_{S,N,j}(t)$$

. where $\tilde{g}_{1,j}$ are calculated from Eq. (13). By using our proposed technique with N $= 9$ and $\nu = 1$ we obtain the following results

$$\tilde{y}_{1,0} = 0.632121, \tilde{y}_{1,1} = -0.310915, \tilde{y}_{1,2} = 0.0514531,$$

$$\tilde{y}_{1,3} = -0.00512502, \tilde{y}_{1,4} = 0.000365153, \tilde{y}_{1,5} = -0.00002025,$$

$$\tilde{y}_{1,6} = 9.3032\times10^{-7}, \tilde{y}_{1,7} = -1.56187\times10^{-8}, \tilde{y}_{1,8} = 1.00461\times10^{-9}, \tilde{y}_{1,9} = -5.90762\times10^{-11}.$$

From Eqs. (41) and (38), $y_2(t)$ is obtained.

Figures 1 and 2, illustrate the behavior of the numerical solutions at N=9 and $\nu = 0.75, 0.85, 0.95$ and 1 with the exact solution of problem (1). At Tables (1) and

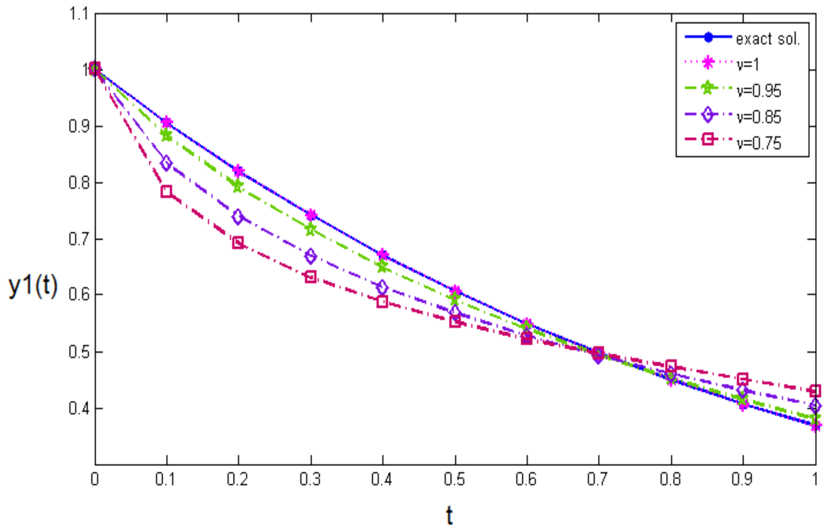

FIGURE 1. The behavior of $y_1(t)$ for $N = 9$ and $\nu = 0.75, 0.85, 0.95, 1$ with the exact solutions of problem (1)

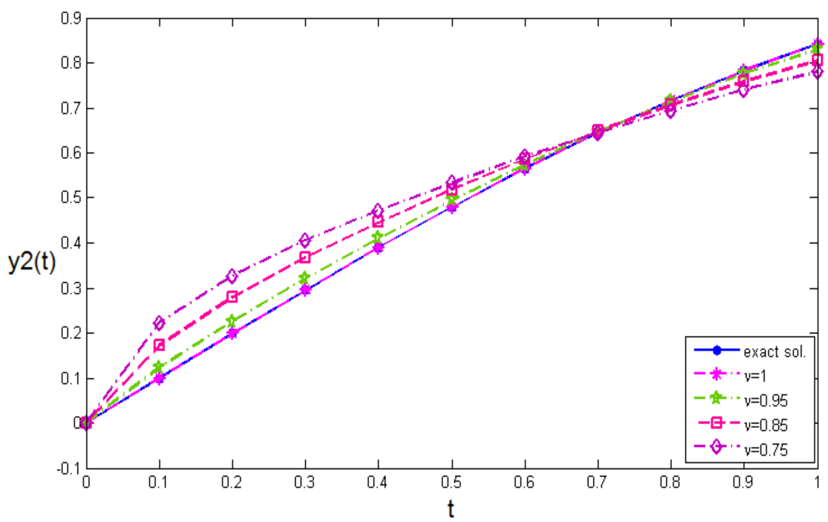

FIGURE 2. The behavior of $y_2(t)$ for $N = 9$ and $\nu = 0.75, 0.85, 0.95, 1$ with the exact solutions of problem (1)

(2), exact and approximate value of $y_1(t)$, $y_2(t)$ for $\nu = 1$ are tabulated in comparison with the results obtained by using ADM, HAM, and VIM methods [21]. It's noted that our approximated results are in a good harmony with the results given in [21]. Also at Tables (3) and (4)the absolute errors of $y_1(t)$ and $y_2(t)$ for problem (1) are calculated at different values of N. It's noted that satisfactory results are obtained by using small numbers of SGPs and the accuracy of our proposed method is increased by using more terms of the polynomial.

t	y_1 exact	y_1 SGOM	y_1 ADM [21]	y_1 VIM [21]	y_1 HAM [21]
0	1	1	1	1	1
0.1	0.904837	0.904837	0.904837	0.904837	0.904837
0.2	0.818731	0.818731	0.818730	0.818730	0.818730
0.3	0.740818	0.740818	0.740818	0.740818	0.740818
0.4	0.67032	0.67032	0.670320	0.670320	0.670320
0.5	0.606531	0.606531	0.6006530	0.606530	0.606530
0.6	0.548812	0.548812	0.548811	0.548811	0.548811
0.7	0.496585	0.496585	0.496585	0.496585	0.496585
0.8	0.449329	0.449329	0.449328	0.449328	0.449328
0.9	0.40657	0.40657	0.406569	0.406569	0.406569
1	0.367879	0.367879	0.367879	0.367879	0.367879

TABLE 1. Numerical results of $y_1(t)$ with comparisons to the results of ADM, HAM and VIM methods [21] for Problem (1) at $\nu = 1$ and N=9.

t	y_2 exact	y_2 SGOM	y_2 ADM [21]	y_2 VIM [21]	y_2 HAM [21]
0	0	-4.38695×10^{-17}	0	0	0
0.1	0.0998334	0.0998334	0.099833	0.099833	0.099833
0.2	0.198669	0.198669	0.198669	0.198669	0.198669
0.3	0.29552	0.29552	0.295520	0.295520	0.295520
0.4	0.389418	0.389418	0.389418	0.389418	0.389418
0.5	0.479426	0.479426	0.479425	0.479425	0.479425
0.6	0.564642	0.564642	0.564642	0.564642	0.564642
0.7	0.644218	0.644218	0.644217	0.644217	0.644217
0.8	0.717356	0.717356	0.717356	0.717356	0.717356
0.9	0.783327	0.783327	0.783326	0.783326	0.783326
1	0.841471	0.841471	0.841471	0.841471	0.841471

TABLE 2. Numerical results of $y_2(t)$ with comparisons to the results of ADM, HAM and VIM methods [21] for Problem (1) at $\nu = 1$ and N=9.

Problem 2 Consider the following non-linear FDAEs [21]

$$D^{\nu} y_1(t) - t y_2^{'}(t) + y_1(t) - (1+t) y_2(t) = 0, \tag{42}$$

$$y_2(t) - \sin(t) = 0, 0 < \nu \leq 1, \tag{43}$$

with initial conditions

$$\begin{aligned} y_1(0) &= 1, \\ y_2(0) &= 0, \end{aligned} \tag{44}$$

with the exact solution is $y_1(t) = e^{-t} + t\sin(t)$, $y_2(t) = \sin(t)$ when $\nu = 1$.

The system can be simplified by substituting Eq. (55) into Eq.(54) to get

$$D^{\nu} y_1(t) + y_1(t) = t\cos(t) + (1+t)\sin(t), \tag{45}$$

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=9)
0	1.11022×10^{-16}	0	0
0.1	3.84005×10^{-4}	1.07213×10^{-6}	8.63101×10^{-9}
0.2	3.74822×10^{-4}	4.19392×10^{-7}	2.05988×10^{-9}
0.3	2.13378×10^{-4}	2.40478×10^{-7}	8.59987×10^{-9}
0.4	5.859×10^{-5}	5.05009×10^{-7}	8.99679×10^{-9}
0.5	4.82868×10^{-6}	6.12651×10^{-7}	8.84285×10^{-11}
0.6	4.06898×10^{-5}	3.09535×10^{-7}	3.48959×10^{-9}
0.7	1.5196×10^{-4}	6.73149×10^{-8}	5.57707×10^{-9}
0.8	2.30822×10^{-4}	4.64×10^{-8}	9.99569×10^{-9}
0.9	1.29377×10^{-4}	5.28447×10^{-7}	7.79981×10^{-9}
1	3.45282×10^{-4}	8.4157×10^{-7}	3.27545×10^{-8}

TABLE 3. The absolute errors of $y_1(t)$ for problem (1) at different values of N.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=9)
0	8.32667×10^{-17}	3.64292×10^{-17}	4.38695×10^{-17}
0.1	3.84005×10^{-4}	1.07213×10^{-6}	8.63101×10^{-9}
0.2	3.74822×10^{-4}	4.19392×10^{-7}	2.05988×10^{-9}
0.3	2.13378×10^{-4}	2.40478×10^{-7}	8.59987×10^{-9}
0.4	5.859×10^{-5}	5.05009×10^{-7}	8.99679×10^{-9}
0.5	4.82868×10^{-6}	6.12651×10^{-7}	8.84285×10^{-11}
0.6	4.06898×10^{-5}	3.09535×10^{-7}	3.48959×10^{-9}
0.7	1.5196×10^{-4}	6.73149×10^{-8}	5.57707×10^{-9}
0.8	2.30822×10^{-4}	4.64×10^{-8}	9.99569×10^{-9}
0.9	1.29377×10^{-4}	5.28447×10^{-7}	7.79981×10^{-9}
1	3.45282×10^{-4}	8.4157×10^{-7}	3.27545×10^{-8}

TABLE 4. The absolute errors of $y_2(t)$ for problem (1) at different values of N.

From Eq. (12) the approximate solution with N=7, is written as

$$y_1(t) = \sum_{j=0}^{7} \tilde{y}_{1,j} C_{S,N,j}^{(\alpha)}(t), \tag{46}$$

and

$$t\cos(t) + (1+t)\sin(t) = \sum_{j=0}^{7} \tilde{g}_{1,j} C_{S,N,j}^{(\alpha)}(t)$$

where $\tilde{g}_{1,j}$ are calculated from Eq. (13). By applying the mechanism described in section(4), at $\nu = 1$ we obtain the following results

$$\tilde{y}_{1,0} = 0.933289, \tilde{y}_{1,1} = 0.125045, \tilde{y}_{1,2} = 0.172925, \tilde{y}_{1,3} = -0.0204415,$$

$$\tilde{y}_{1,4} = -0.00154787, \tilde{y}_{1,5} = 0.0000723288, \tilde{y}_{1,6} = 8.38502\times10^{-6}, \tilde{y}_{1,7} = -3.22501\times10^{-7}.$$

t	y_1 exact	y_1 SGOM	y_1 ADM [21]	y_1 HAM [21]	y_1 VIM [21]
0	1	1	1	1	1
0.1	0.914821	0.914821	0.914820	0.914820	0.914820
0.2	0.858465	0.858465	0.858464	0.858464	0.858464
0.3	0.829474	0.829474	0.829474	0.829474	0.829474
0.4	0.826087	0.826087	0.826087	0.826087	0.826087
0.5	0.846243	0.846243	0.846243	0.846243	0.846243
0.6	0.887597	0.887597	0.887597	0.887597	0.887597
0.7	0.947538	0.947538	0.947537	0.947537	0.947537
0.8	1.02321	1.02321	1.023213	1.023213	1.023213
0.9	1.11156	1.11156	1.111563	1.111563	1.111563
1	1.20935	1.20935	1.209350	1.209350	1.209350

TABLE 5. Numerical results of $y_1(t)$ with comparisons to the results of ADM, HAM and VIM methods [21] for Problem (2) at $\nu = 1$.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=7)
0	0	1.11022×10^{-16}	1.11022×10^{-16}
0.1	1.77724×10^{-3}	1.07518×10^{-5}	8.59441×10^{-8}
0.2	1.8681×10^{-3}	5.24695×10^{-6}	3.95822×10^{-8}
0.3	1.25709×10^{-3}	3.79883×10^{-6}	6.30974×10^{-8}
0.4	6.23628×10^{-4}	6.42335×10^{-6}	7.27392×10^{-8}
0.5	3.49004×10^{-4}	7.66675×10^{-6}	4.27825×10^{-8}
0.6	5.2971×10^{-4}	5.06614×10^{-6}	2.26003×10^{-8}
0.7	9.96761×10^{-4}	1.58552×10^{-6}	4.21027×10^{-8}
0.8	1.34066×10^{-3}	2.43485×10^{-6}	5.56943×10^{-8}
0.9	9.41495×10^{-4}	6.74133×10^{-6}	1.34622×10^{-8}
1	9.96328×10^{-4}	5.40103×10^{-6}	9.14367×10^{-8}

TABLE 6. The absolute errors of $y_1(t)$ for problem (2) at different values of N.

In Figure 3, the approximate results of $y_1(t)$ at $N = 7$ are plotted for $\nu = 0.75, 0.85, 0.95$ and 1 with the exact solution. It's noted that our approximate solutions covers the classical results as the fractional derivatives goes to unity. In Table (5), the exact solution for $\nu = 1$ and approximate values of $y_1(t)$ are shown with comparisons by ADM, HAM and VIM methods [21]. The results are in a good harmony with the results given in [21]. In Table (6), The absolute errors of $y_1(t)$ for problem (2) at different values of N are calculated. It's observed that the efficiency of our proposed method is increased by increasing N.

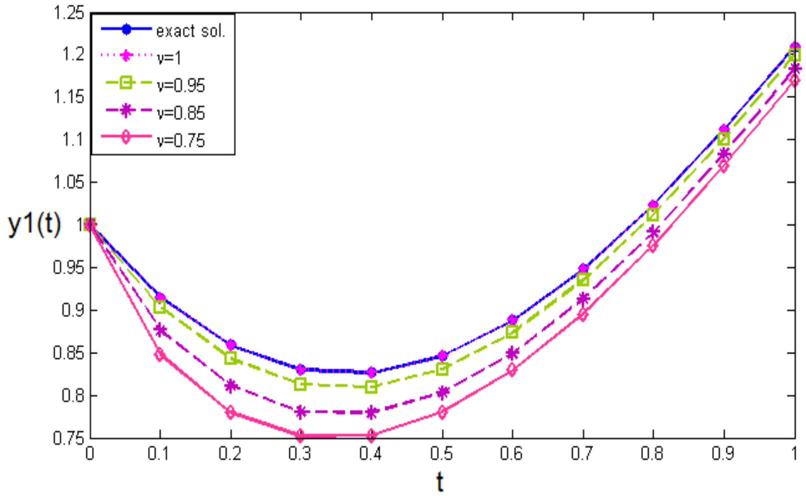

FIGURE 3. The behavior of $y_1(t)$ for $N = 7$ and $\nu = 0.75, 0.85, 0.95, 1$ with the exact solutions of problem (2)

Problem 3 Consider the following non-linear FDAEs [21]

$$D^{\nu_1}y_1(t) - ty_2^{'}(t) + t^2y_3^{'}(t) + y_1(t) - (1+t)y_2(x) + (t^2+2t)y_3(t) = 0, \tag{47}$$

$$D^{\nu_2}y_2(t) - ty_3^{'}(t) - y_2(t) + (t-1)y_3(t) = 0, \tag{48}$$

$$y_3(t) - \sin(t) = 0, 0 < \nu \leq 1, \tag{49}$$

with initial conditions

$$\begin{aligned} y_1(0) &= 1, \\ y_2(0) &= 1, \\ y_3(0) &= 0, \end{aligned} \tag{50}$$

and the exact solution is $y_1(t) = e^{-t} + te^t$,$y_2(t) = e^t + t\sin(t)$,$y_3(t) = \sin(t)$ at $\nu = 1$. By substituting Eq. (49) into Eqs. (49) and Eq. (47), the problem is converted to the following system

$$D^{\nu_1}y_1(t) - ty_2^{'}(t) + t^2\cos(t) + y_1(t) - (1+t)y_2(t) + (t^2+2t)y_3(t) = 0, \tag{51}$$

$$D^{\nu_2}y_2(t) - t\cos(t) - y_2(t) + (t-1)\sin(t) = 0, \tag{52}$$

which is easy to solve.

t	y_1 exact	y_1 SGOM	y_1 ADM [21]	y_1 HAM [21]	y_1 VIM [21]
0	1	1	1	1	1
0.1	1.01535	1.01535	1.01535	1.01535	1.01535
0.2	1.06301	1.06301	1.06301	1.06301	1.06301
0.3	1.14578	1.14578	1.14577	1.14577	1.14577
0.4	1.26705	1.26705	1.26704	1.26705	1.26704
0.5	1.43089	1.43089	1.43089	1.43089	1.43089
0.6	1.64208	1.64208	1.64208	1.64208	1.64208
0.7	1.90621	1.90621	1.90621	1.90621	1.90621
0.8	2.22976	2.22976	2.22976	2.22975	2.22976
0.9	2.62021	2.62021	2.62021	2.62019	2.62021
1	3.08616	3.08616	3.08616	3.08613	3.08616

Table 7. Numerical results of $y_1(t)$ with comparisons to the results of ADM, HAM and VIM methods [21] for Problem (3) at $\nu = 1$.

t	y_2 exact	y_2 SGOM	y_2 ADM [21]	y_2 HAM [21]	y_2 VIM [21]
0	1	1	1	1	1
0.1	1.11515	1.11515	1.11515	1.11515	1.11515
0.2	1.26114	1.26114	1.26113	1.26113	1.26113
0.3	1.43851	1.43851	1.43851	1.43851	1.43851
0.4	1.64759	1.64759	1.64759	1.64759	1.64759
0.5	1.88843	1.88843	1.88843	1.88843	1.88843
0.6	2.16090	2.16090	2.16090	2.16090	2.16090
0.7	2.46471	2.46471	2.46470	2.46470	2.46470
0.8	2.79943	2.79943	2.79942	2.79943	2.79942
0.9	3.1646	3.1646	3.16459	3.16460	3.16459
1	3.55975	3.55975	3.55975	3.55975	3.55975

Table 8. Numerical results of $y_2(t)$ with comparisons to the results of ADM, HAM and VIM methods [21] for Problem (3) at $\nu = 1$.

Figures 4 and 5, show the approximate results of $y_1(t)$ and $y_2(t)$ of problem (3) at $N = 7$ and $\nu = 0.75, 0.85, 0.95$ and 1 with the exact solution of problem (3). In Tables (7) and (8), the exact solution and the approximated results of $y_1(t) and$ $y_2(t)$ respectively are shown with comparisons to ADM, HAM and VIM methods [21]. The obtained results are in good agreement with the results given in [21]. In Tables (9) and (10), the absolute errors of $y_1(t)$ and $y_2(t)$ for problem (3) at different values of N are calculated.

Problem 4 Consider the following non-linear FDAEs [23]

$$D^{0.5}y_1(t) + 2y_1(t) - \frac{\Gamma(7/2)}{\Gamma(3)}y_2(t) + y_3(t) = 2t^{5/2} + \sin(t), \tag{53}$$

$$D^{0.5}y_2(t) + y_2(t) + y_3(t) = \frac{\Gamma(3)}{\Gamma(5/2)}t^{3/2} + t^2 + \sin(t), \tag{54}$$

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=7)
0	2.22045×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}
0.1	4.97105×10^{-3}	1.92607×10^{-5}	4.39611×10^{-7}
0.2	5.30764×10^{-3}	1.07711×10^{-5}	1.78988×10^{-7}
0.3	3.46418×10^{-3}	1.0193×10^{-5}	3.03422×10^{-7}
0.4	1.29887×10^{-3}	1.75627×10^{-5}	3.5295×10^{-7}
0.5	1.37983×10^{-5}	2.24578×10^{-5}	1.60322×10^{-7}
0.6	8.59475×10^{-5}	1.98912×10^{-5}	1.57824×10^{-8}
0.7	1.18782×10^{-3}	1.59523×10^{-5}	1.26465×10^{-7}
0.8	2.09676×10^{-3}	2.20894×10^{-5}	2.05662×10^{-7}
0.9	5.91674×10^{-4}	3.67764×10^{-5}	1.19601×10^{-7}
1	6.66435×10^{-3}	1.31455×10^{-5}	4.42899×10^{-7}

TABLE 9. The absolute errors of $y_1(t)$ for problem (3) at different values of N.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=7)
0	0	0	2.22045×10^{-16}
0.1	1.49468×10^{-3}	1.56747×10^{-5}	9.0206×10^{-8}
0.2	1.7786×10^{-3}	1.11145×10^{-5}	6.33029×10^{-8}
0.3	1.59374×10^{-3}	1.11285×10^{-5}	9.4408×10^{-8}
0.4	1.41731×10^{-3}	1.63529×10^{-5}	1.18315×10^{-7}
0.5	1.48943×10^{-3}	2.03916×10^{-5}	1.11268×10^{-7}
0.6	1.84769×10^{-3}	1.9978×10^{-5}	1.09519×10^{-7}
0.7	2.36861×10^{-3}	1.82895×10^{-5}	1.38119×10^{-7}
0.8	2.81571×10^{-3}	2.15997×10^{-5}	1.66572×10^{-7}
0.9	2.8941×10^{-3}	2.95066×10^{-5}	1.56955×10^{-7}
1	2.3111×10^{-3}	1.9018×10^{-5}	2.26059×10^{-7}

TABLE 10. The absolute errors of $y_2(t)$ for problem (3) at different values of N.

$$2y_1(t) + y_2(t) - y_3(t) = 2t^{5/2} + t^2 - \sin(t), t \in [0, 1], \tag{55}$$

with initial conditions

$$y_1(0) = y_2(0) = y_3(0) = 0. \tag{56}$$

The exact solution of this problem is

$$y_1(t) = t^{5/2},$$

$$y_2(t) = t^2,$$

$$y_3(t) = \sin(t).$$

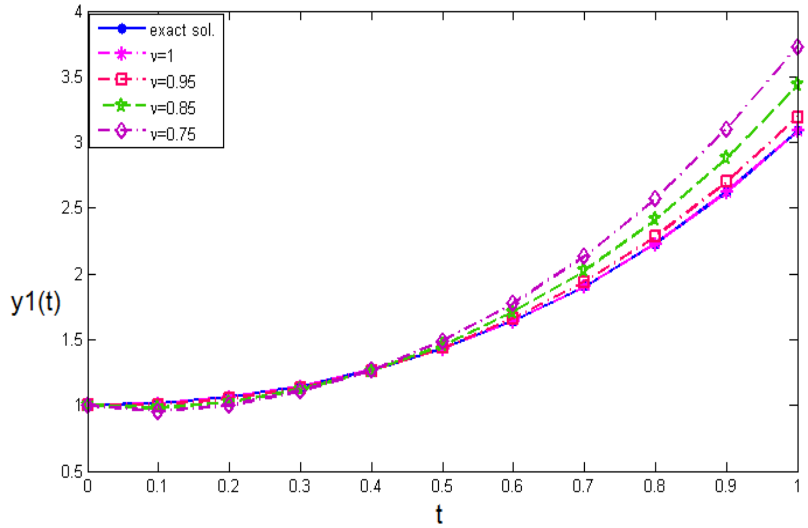

FIGURE 4. The behavior of $y_1(t)$ for $N = 7$ and $\nu = 0.75, 0.85, 0.95, 1$ with the exact solutions of problem (3)

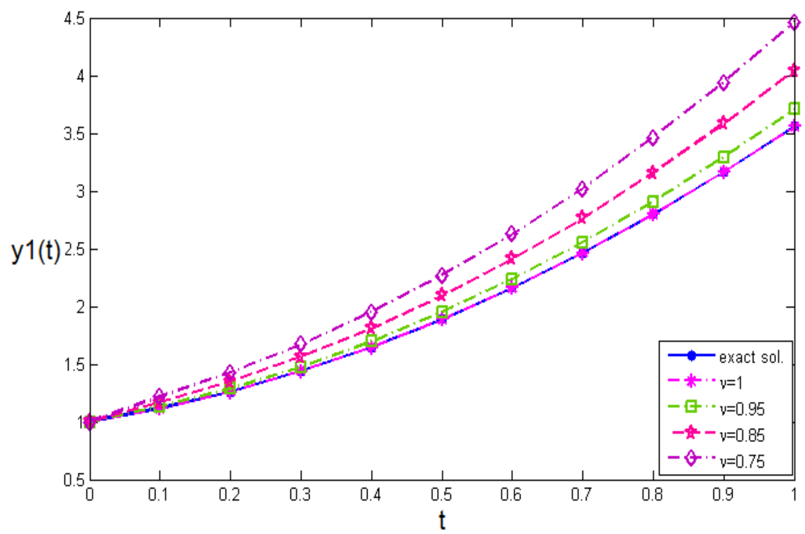

FIGURE 5. The behavior of $y_2(t)$ for $N = 7$ and $\nu = 0.75, 0.85, 0.95, 1$ with the exact solutions of problem (3)

In Tables (11)- (13), the absolute errors of $y_1(t)$, $y_2(t)$ and $y_3(t)$ for problem (4) at different values of N are calculated.

Problem 5 Find the solution of the following linear initial value problem,

$$D^{\nu} y_1(t) + 5y_2(t) = 0, 0 < \nu < 1 \tag{57}$$

$$y_2(t) = \frac{1}{5} y_1(t), \tag{58}$$

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=9)
0	0	1.30104×10^{-17}	1.48536×10^{-17}
0.1	2.3147×10^{-3}	1.66177×10^{-4}	3.63967×10^{-6}
0.2	1.83998×10^{-3}	3.60104×10^{-5}	6.72661×10^{-6}
0.3	7.94018×10^{-4}	4.53895×10^{-5}	2.35059×10^{-6}
0.4	8.99177×10^{-5}	9.42631×10^{-5}	$3.74093t \times 10^{-6}$
0.5	2.42102×10^{-5}	9.10098×10^{-5}	4.23489×10^{-6}
0.6	5.07183×10^{-4}	3.86001×10^{-5}	1.49777×10^{-6}
0.7	1.177×10^{-3}	4.55973×10^{-6}	2.90314×10^{-6}
0.8	1.46647×10^{-3}	4.36063×10^{-5}	3.3113×10^{-6}
0.9	6.46246×10^{-4}	9.66677×10^{-5}	1.0196×10^{-6}
1	2.14534×10^{-3}	1.23426×10^{-4}	3.84292×10^{-6}

TABLE 11. The absolute errors of $y_1(t)$ for problem (4) at different values of N.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=9)
0	0	3.67917×10^{-17}	1.34043×10^{-18}
0.1	2.45591×10^{-4}	3.41285×10^{-5}	2.75641×10^{-6}
0.2	3.66087×10^{-4}	3.52352×10^{-5}	3.1381×10^{-6}
0.3	3.93023×10^{-4}	2.92161×10^{-5}	2.93947×10^{-6}
0.4	3.57935×10^{-4}	2.68372×10^{-5}	1.93389×10^{-6}
0.5	2.9236×10^{-4}	2.87508×10^{-5}	2.56966×10^{-6}
0.6	2.27833×10^{-4}	3.05117×10^{-5}	2.72018×10^{-6}
0.7	1.95891×10^{-4}	2.75936×10^{-5}	1.44028×10^{-6}
0.8	2.28071×10^{-4}	2.04057×10^{-5}	2.18284×10^{-6}
0.9	3.55907×10^{-4}	1.93086×10^{-5}	2.21861×10^{-6}
1	6.10937×10^{-4}	4.96308×10^{-5}	6.07829×10^{-6}

TABLE 12. The absolute errors of $y_2(t)$ for problem (4) at different values of N.

with the initial conditions

$$y_1(0) = 1, y_2(0) = \frac{1}{5}. \tag{59}$$

The exact solution of this problem is [30]

$$y_1(t) = \sum_{k=0}^{\infty} \frac{(-t^\nu)^k}{\Gamma(\nu k + 1)}.$$

By substituting Eq. (58) into Eq. (57), the system is converted to the following system

$$D^\nu y_1(t) + y_1(t) = 0, \tag{60}$$

which is easy to solve.

Table (14) illustrate absolute errors comparison of $y_1(t)$ for $N = 10$ and different values of $\nu = 0.2, 0.4, 0.8$ and 1 between our proposed mechanism and the method mentioned in [30]. From Table 14, we see that as ν approaches an integer value the error is reduced, as predicted.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=9)
0	0	6.28126×10^{-17}	3.10476×10^{-17}
0.1	4.3838×10^{-3}	2.98225×10^{-4}	4.52292×10^{-6}
0.2	3.31387×10^{-3}	3.67855×10^{-5}	1.03151×10^{-5}
0.3	1.19501×10^{-3}	6.15628×10^{-5}	1.76171×10^{-6}
0.4	1.781×10^{-4}	1.61689×10^{-4}	5.54797×10^{-6}
0.5	2.43939×10^{-4}	1.53269×10^{-4}	5.90013×10^{-6}
0.6	7.86534×10^{-4}	4.66884×10^{-5}	2.75369×10^{-7}
0.7	2.1581×10^{-3}	1.84742×10^{-5}	4.36601×10^{-6}
0.8	2.70486×10^{-3}	6.68068×10^{-5}	4.43977×10^{-6}
0.9	9.36585×10^{-4}	1.74027×10^{-4}	1.79411×10^{-7}
1	4.90162×10^{-3}	2.96482×10^{-4}	1.37641×10^{-5}

TABLE 13. The absolute errors of $y_3(t)$ for problem (4) at different values of N.

	$\nu = 0.2$		$\nu = 0.4$	
t	y_1 LOM [30]	y_1 SGOM	y_1 LOM [30]	y_1 SGOM
0.1	2.9×10^{-1}	1.9×10^{-1}	3.9×10^{-1}	6.9×10^{-2}
0.3	4.5×10^{-1}	1.7×10^{-1}	5.1×10^{-1}	5.7×10^{-2}
0.5	7.4×10^{-1}	1.6×10^{-1}	7.3×10^{-1}	5.2×10^{-2}
0.7	3.7×10^{-1}	1.6×10^{-1}	3.3×10^{-1}	5.0×10^{-2}
0.9	2.0×10^{-1}	1.6×10^{-1}	2.2×10^{-1}	4.8×10^{-2}
	$\nu = 0.8$		$\nu = 1$	
t	y_1 LOM [30]	y_1 SGOM	y_1 SGOM	
0.1	1.1×10^{-3}	3.9×10^{-3}	1.5×10^{-9}	
0.3	2.1×10^{-4}	3.1×10^{-3}	1.5×10^{-9}	
0.5	8.4×10^{-4}	2.6×10^{-3}	2.1×10^{-9}	
0.7	8.7×10^{-4}	2.4×10^{-3}	1.0×10^{-9}	
0.9	5.8×10^{-4}	2.1×10^{-3}	1.6×10^{-10}	

TABLE 14. The absolute errors of $y_1(t)$ at $N = 10$ for Problem (5) at different values of ν in comparison with the results in [30].

7. Conclusions

In this paper, A new numerical mechanism has been derived to find the approximate solutions of the FADEs, which depends on the SGOM of fractional integration. The proposed mechanism depends on the shifted Gegenbauer and Tau method. The applicability, accuracy and rapidity by using few terms of the SGPs of the proposed mechanism are illustrated by numerical problems. Numerical comparisons with other methods in the literature are held which demonstrate the efficiency of our proposed method.

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Hoda F. Ahmed
Minia University, Faculty of Science, Department of Mathematics, Minia, Egypt
E-mail address: hodamina@yahoo.com

Marina B. Melad
Assuit University Branch of New- Vally, Faculty of Science, Department of Mathematics, Assuit, Egypt
E-mail address: marina.benyamin.melad@gmail.com