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ON RIESZ-CAPUTO FRACTIONAL DIFFERENTIATION MATRIX OF RADIAL BASIS FUNCTIONS VIA COMPLEX STEP DIFFERENTIATION METHOD

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ABSTRACT. In literature, evaluation of the fractional differentiation matrix for implementation of radial basis function methods is often achieved via Taylor series expansion. Term by term fractional differentiation of expanded functions usually leads to algebraic complexities and round up errors. This paper focuses on a simplified scheme that evaluates Riesz-Caputo fractional derivative of radial basis functions by applying the complex step differentiation technique. The numerical test example has shown that this approach is more effective and accurate. Furthermore, we solved a two dimensional space fractional system with symmetric Riesz-Caputo fractional derivative using hybrid radial basis function pseudospectral where the fractional differentiation matrices are used for implementation.

1. INTRODUCTION

Consider the following space-time fractional partial differential equation (STFPDE)

$$\partial_t^\alpha u = D_C \partial_{|x|}^\beta u + G(t, x, u), \qquad\qquad x \in \Omega, \ t > 0 \qquad (1)$$

$$u(x, 0) = u_0(x), \quad x \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0, \quad x \text{ on } \partial\Omega$$

where ∂_t^α denotes the left side Caputo derivative for $0 < \alpha < 1$, ${}_C\partial_{|x|}^\beta = \lambda {}_a^C\partial_x^\beta + (1 - \lambda) {}_x^C\partial_b^\beta$ is Riesz-Caputo fractional derivative with $1 < \beta < 2$ and $\lambda \in [0, 1]$ is the skewness parameter. System (1) is suitable for modelling natural phenomena that are observed to exhibit anomalous diffusion [1, 2, 3, 4, 5, 7, 8] with respect to time or space or both. Most of the models that describe real life observations or experiments carried out on irregular domains require a meshless technique for interpolation. Consequently, the accuracy and computational cost effectiveness of the radial basis function (RBF) methods will continue to draw the attention of researchers. Apart from accuracy of the RBF techniques, their applicability in solving problems modelled on irregular domains make them to stand out amongst other numerical methods such as finite difference methods, finite element methods and finite volume methods [9, 12, 18].

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In this paper, an effective RBF discretization method of the operator $_C\partial_{|x|}^\beta$ using complex step differentiation (CSD) technique is presented. The standard approach of the RBF methods is that, for a set of data points $x_k \in \mathbb{R}^n$, $k = 1, \ldots, N$, then the radial basis interpolant takes the form

$$u_N(x) = \sum_{k=1}^{N} \eta_k \psi(\|x - x_{ik}\|),$$

where $\|.\|$ is the Euclidean norm and η_k are the coefficients. The required Riesz-Caputo fractional derivative of the RBF is given as

$_C\partial_{|x|}^\beta[u_N(x)] = \sum_{k=1}^{N} \eta_{kC}\partial_{|x|}^\beta[\psi(\|x - x_k\|)],$

and can be expressed in the matrix form as

$$\Longrightarrow U = A\eta, \qquad U^\beta = A_x^\beta \eta, \Longrightarrow U^\beta = A_x^\beta A^{-1}\eta, \tag{2}$$

where $A_{ik} = \psi(\|x_i - x_k\|)$, $A_{ik}^\beta = {}_C\partial_{|x|}^\beta \psi \|x_i - x_k\|$, $\eta = [\eta_1 \ldots \eta_N]^T$,

$U = [u_1 \ldots u_N]^T$, $i, k = 1 \ldots N$. In literature, the entries of the matrix A_{ik}^β are often obtained by expanding the RBFs via Taylor series expansion and differentiating term by term. For instance, in work of Mohammadi and Schaback [13], the fractional derivatives of the familiar Gaussian (GA), Multiquadrics (MQ) and several other RBFs were obtained through term by term differentiation of Maclaurin series. Piret and Hanert [16] also followed the same approach to discretise a Riesz operator in solving a fractional diffusion model. Alternatively, Pang et al. [14] solved a fractional advection–dispersion equation using RBF by applying the Gauss–Jacobi quadrature to evaluate A_{ik}^β. These procedures increase the computational cost and are liable to truncation error. In the light of the above, we will start with basic definitions and relevant concepts before presenting CSD method.

2. Basic Concepts

Definition 1. [17]: The Riesz-Caputo and Riesz- Riemann-Liouville fractional derivatives of order $\beta \in \mathbb{R}^+$, $n - 1 < \beta < n$, $n \in \mathbb{N}$ of a function $v(x, t)$ denoted by $_C\partial_{|x|}^\beta v(x, t)$ and $_R\partial_{|x|}^\beta v(x, t)$ are defined by

$$_C\partial_{|x|}^\beta = \lambda_a^C\partial_x^\beta + (1 - \lambda)_x^C\partial_b^\beta, \quad _R\partial_{|x|}^\beta = \lambda_a^R\partial_x^\beta + (1 - \lambda)_x^R\partial_b^\beta, \tag{3}$$

where $\lambda \in [0, 1]$ is the skewness parameter and

$$_a^C\partial_x^\beta v(x, t) = \frac{1}{\Gamma(n - \beta)} \int_a^x \frac{\partial^n v(\tilde{x}, t)}{\partial \tilde{x}^n}(x - \tilde{x})^{n-\beta-1}d\tilde{x}, \ \beta > 0, \ x > a, \tag{4}$$

$$_x^C\partial_b^\beta v(x, t) = \frac{(-1)^n}{\Gamma(n - \beta)} \int_x^b \frac{\partial^n v(\tilde{x}, t)}{\partial \tilde{x}^n}(\tilde{x} - x)^{n-\beta-1}d\tilde{x}, \beta > 0, \ b > x, \tag{5}$$

and

$$_a^R\partial_x^\beta v(x, t) = \frac{1}{\Gamma(n - \beta)} \frac{\partial^n}{\partial x^n} \int_a^x v(\tilde{x}, t)(x - \tilde{x})^{n-\beta-1}d\tilde{x}, \ \beta > 0, \ x > a, \tag{6}$$

$$_x^R\partial_b^\beta v(x, t) = \frac{(-1)^n}{\Gamma(n - \beta)} \frac{\partial^n}{\partial x^n} \int_x^b v(\tilde{x}, t)(\tilde{x} - x)^{n-\beta-1}d\tilde{x}, \ \beta > 0, \ b > x. \tag{7}$$

Definition 2. [17]: Applying integration by parts, equivalent definition of Caputo fractional derivatives of order $\beta \in \mathbb{R}^+$, $1 < \beta < 2$, of a function $v(x)$ are given thus

$$_{a}^{C}\partial_{x}^{\beta}v(x) = \frac{(x-a)^{2-\beta}}{\Gamma(3-\beta)}v''(a) + \frac{1}{\Gamma(3-\beta)}\int_{a}^{x}v'''(\tilde{x})(x-\tilde{x})^{2-\beta}d\tilde{x}, \tag{8}$$

$$_{x}^{C}\partial_{b}^{\beta}v(x) = \frac{(b-x)^{2-\beta}}{\Gamma(3-\beta)}v''(b) + \frac{1}{\Gamma(3-\beta)}\int_{x}^{b}v'''(\tilde{x})(\tilde{x}-x)^{2-\beta}d\tilde{x}. \tag{9}$$

Definition 3. [6] Radial basis function is a real valued function $\psi(r,q)$ where

$$r_i = \|\mathbf{x} - \mathbf{x}_i\|, \quad \mathbf{x}\in\mathbb{R}^n \tag{10}$$

where q is the shape parameter and r represents the Euclidean distance from a set of centres ($\mathbf{x}_i$) to data points ($\mathbf{x}$). The famous Gaussian (GA), Multiquadric (MQ), Cubic (CB) are given below respectively

$$\psi(r,\ q) = e^{-(\ qr)^2}, \psi(r,\ q) = \sqrt{q^2 + r^2}, \ \ \psi(r) = r^3. \tag{11}$$

3. Complex Step Differentiation Method (CSDM)

Let $v(x)$ be continuous a function on $[a,b]$ which is infinitely differentiable and extendable into the complex plane, then we have the Taylor series

$$v(x + \epsilon) = v(x) + \epsilon v'(x) + \frac{\epsilon^2}{2!}v''(x) + \cdots + \frac{\epsilon^n}{n!}v^{(n)}(x) + \ldots. \tag{12}$$

If we replace the step size ϵ by $i\epsilon$ where $i = \sqrt{-1}$, we obtain

$$v(x + i\epsilon) = v(x) + i\epsilon v'(x) - \frac{\epsilon^2}{2!}v''(x) + \cdots + \frac{(i\epsilon)^n}{n!}v^{(n)}(x) + \ldots, \tag{13}$$

by taking the imaginary part of (13), we have

$$\text{Im}\{v(x + i\epsilon)\} = \epsilon v'(x) - \frac{\epsilon^3}{3!}v'''(x) + \frac{\epsilon^5}{5!}v^{(5)}(x) + \ldots. \tag{14}$$

Consequently, the CDS approximate for first order derivative with a specific step size ϵ_1 is given by

$$v'(x) \approx \frac{\text{Im}\{v(x + i\epsilon_1)\}}{\epsilon_1} + O\left(\epsilon_1^2\right). \tag{15}$$

In the work of [10], the second order approximation was obtained using

$$\zeta = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \quad \xi = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

thus

$$v''(x) \approx \frac{\text{Im}\{v(x + \xi\epsilon_2) + v(x - \xi\epsilon_2)\}}{\epsilon_2^2} + O\left(\epsilon_2^2\right), \tag{16}$$

and performing the operation $v(x - i\epsilon_3) - v(x + i\epsilon_3)$ we realise the third order approximation

$$v'''(x) \approx \frac{3\text{Im}\{v(x - i\epsilon_3) - v(x + i\epsilon_3)\} + 6\epsilon\text{Im}\left\{\frac{v(x+i\epsilon_1)}{\epsilon_1}\right\}}{\epsilon_3^3} + O\left(\epsilon_3^2\right). \tag{17}$$

Now, using equations (15)-(17) and in view Definition 2, the symmetric Riesz-Caputo can be approximated thus

$$
\begin{aligned}
{}_{C}\partial^{\beta}_{|t|}\psi(t) &= \frac{(\tau-a)^{2-\beta}\text{Im}\{\psi(\tau+\xi\epsilon_2)+\psi(\tau-\xi\epsilon_2)\}}{2\Gamma(3-\beta)\epsilon_2^2} \\
&+ \frac{0.5}{\Gamma(3-\beta)}\int_a^t \frac{\left(3\text{Im}\{\psi(\tau-i\epsilon_3)-\psi(\tau+i\epsilon_3)\}+6\epsilon\text{Im}\left\{\frac{\psi(\tau+i\epsilon_1)}{\epsilon_1}\right\}\right)d\tau}{\epsilon_3^3(t-\tau)^{\beta-2}} \\
&+ \frac{(b-\tau)^{2-\beta}\text{Im}\{\psi(\tau+\xi\epsilon_2)+\psi(\tau-\xi\epsilon_2)\}}{2\Gamma(3-\beta)\epsilon_2^2} \\
&+ \frac{0.5}{\Gamma(3-\beta)}\int_t^b \frac{\left(3\text{Im}\{\psi(\tau-i\epsilon_3)-\psi(\tau+i\epsilon_3)\}+6\epsilon\text{Im}\left\{\frac{\psi(\tau+i\epsilon_1)}{\epsilon_1}\right\}\right)d\tau}{\epsilon_3^3(\tau-t)^{\beta-2}}.
\end{aligned} \tag{18}
$$

Remark 1. For $1 < \beta < 2$, the kernels of left and right Caputo derivatives are switched from $(t-\tau)^{1-\beta}$ to $(t-\tau)^{2-\beta}$ and $(\tau-t)^{1-\beta}$ to $(\tau-t)^{2-\beta}$ respectively as in Definition 2 by virtue of integration by parts and then used for approximation formula (18) in order to control the growth of the singularity of the integrals so that as $\beta \to 2$, $(2-\beta) \to 0$. Also for the accuracy of the approximation, we used different step sizes for first, second and third CSD formulas.

4. APPLICATION

4.1. **Test Example:** We use the CB basis function with centre at the origin to test accuracy of the approximation formula presented in (18). Using a famous result for $\lambda = 1$, $x \in [0, 1]$ we have

$$
{}_{C}\partial^{\beta}_{|x|}\left(x^3\right) = {}_{0}^{C}\partial^{\beta}_{x}\left(x^3\right) = \frac{\Gamma(4)\,x^{3-\beta}}{\Gamma(4-\beta)} \tag{19}
$$

We used $\epsilon_1 = 1.110223 \times 10^{-16}$, $\epsilon_2 = 10^{-2}$, and $\epsilon_3 = 10^{-4}$. The asymmetric Riesz-Caputo fractional derivative is presented in the Table 1 below with absolute errors.

TABLE 1. The exact and approximate values of Riesz-Caputo of x^3 for $\beta = 1.5$ with absolute errors

x	Exact Values	Approximate Values	Absolute error
0.01	0.00117969066501236	0.00117969066501144	9.24607612695638E-16
0.02	0.00396799060526867	0.00396799060526416	4.50074005842183E-15
0.03	0.00806733958775807	0.00806733958774907	8.99454122293975E-15
0.04	0.01334667630292340	0.01334667630291940	3.97078203651091E-15
0.05	0.01972266737745510	0.01972266737746220	7.09154956979319E-15

4.2. **Space fractional virus within host model:** Consider the following system of equations [15]:

$$u_t = D_1 \,_C\partial^\beta_{|x|} u + a - buv - d_1 u, \quad x, y \in \Omega, \ t > 0 \tag{20}$$

$$w_t = D_2 \,_C\partial^\beta_{|x|} w + buv - d_2 w, \quad x, y \in \Omega, \ t > 0 \tag{21}$$

$$v_t = D_3 \,_C\partial^\beta_{|x|} v + cw - d_3 v, \quad x, y \in \Omega, \ t > 0 \tag{22}$$

$$u(x, y, 0) = 1000 e^{-\frac{1}{2}(x^2 + y^2)}, \quad w(x, y, 0) = 0,$$

$$v(x, y, 0) = 1, \ \text{in } \Omega, \ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $u(x, y, t)$ is uninfected target cells, $w(x, y, t)$ the infected cells, and $v(x, y, t)$ the free virus, D_1, D_2, D_3 are diffusion coefficients, a is the target cells production rate, b is the infection rate, c free virus production rate by infected cells, d_1, d_2, d_3 are the death rates. System (20)-(22) above is the modified version of the famous target cell limited of viral infections [11, 19, 21, 22]. Here, the nonlinear space derivatives are added to capture anomalous transport phenomena observed with respect to the space variable.

Now, we apply the famous RBF-Pseudospectral technique to effect the solution using (20)-(22) so that, the time derivative is discretized using the Euler forward difference method such for $t^n = nh$,

$$\begin{aligned} u^{n+1} - u^n &= h u_t, \\ w^{n+1} - w^n &= h w_t, \\ v^{n+1} - v^n &= h v_t, \end{aligned} \tag{23}$$

while the space variables will be interpolated at collocation points $\{x_i\}_{i=1}^N$ for $x \in \mathbb{R}^2$ thus

$$u^n(x) = \sum_{k=1}^N \eta^n_k \psi(\|x - x_k\|),$$

$$w^n(x) = \sum_{k=1}^N \alpha^n_k \psi(\|x - x_k\|),$$

$$v^n(x) = \sum_{k=1}^N \rho^n_k \psi(\|x - x_k\|).$$

Now, we express the following matrices thus,

$$\mathbf{U}^n = \{u^n(x)\}_{k=1}^N, \quad \mathbf{W}^n = \{w^n(x)\}_{k=1}^N, \quad \mathbf{V}^n = \{v^n(x)\}_{k=1}^N.$$

Let, $\mathbf{U}^n_{|x|} = A^\beta_{ik} A^{-1}_{ik} \mathbf{U}^n$, $\mathbf{W}^n_{|x|} = A^\beta_{ik} A^{-1}_{ik} \mathbf{W}^n$ and $\mathbf{V}^n_{|x|} = A^\beta_{ik} A^{-1}_{ik} \mathbf{V}^n$ then, we have the scheme

$$\mathbf{U}^{n+1} - \mathbf{U}^n = h \left(D_1 \mathbf{U}^n_{|x|} + a - b\mathbf{U}^n * \mathbf{V}^n - d_1 \mathbf{U}^n \right), \tag{24}$$

$$\mathbf{W}^{n+1} - \mathbf{W}^n = h \left(D_2 \mathbf{W}^n_{|x|} + b\mathbf{U}^n * \mathbf{V}^n - d_2 \mathbf{W}^n \right), \tag{25}$$

$$\mathbf{V}^{n+1} - \mathbf{V}^n = h \left(D_3 \mathbf{V}^n_{|x|} + c\mathbf{W}^n - d_3 \mathbf{V}^n \right). \tag{26}$$

Remark 2: We used the time step size h small enough to ensure the stability of the scheme; the symbol * is component-wise multiplication of matrices. More also, we set the shape parameter $q = 0.4$ and number of data point $N = 20$ to avoid ill-conditioning challenges associated with RBF methods.

The numerical solution of the system (20)–(22) is shown respectively for uninfected target cells, infected cells, and free virons in Figure 1(A)–1(C), Figure 2(A)–2(C) and Figure 3(A)–3(C) for $\beta = $ 1.7 at $t = 0.1, \quad 0.2, \quad 0.3$. We choose $h = 0.005$, and the parameter values in [20]: $a = 10, \quad b = 0.33, \quad c = 5, \quad d_1 = 0.02, \quad d_2 = 0.24, \quad d_3 = 2.4, \quad D_1 = 0.65, \quad D_2 = 0.65, \quad D_3 = 0.12$. The numerical computations were done using MATLAB software.

(A) $t = 0.1$ (B) $t = 0.2$ (C) $t = 0.3$
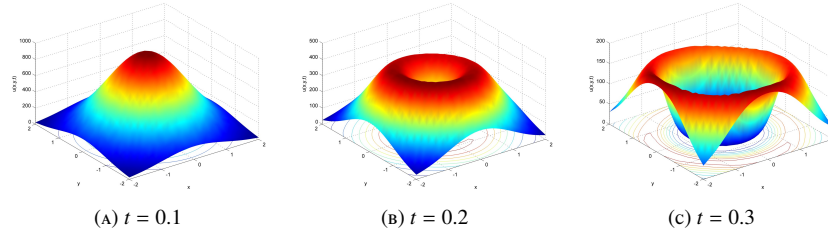
FIGURE 1. Density of uninfected target cells

(A) $t = 0.1$ (B) $t = 0.2$ (C) $t = 0.3$
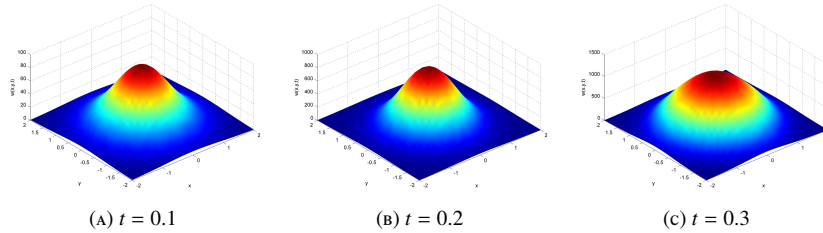
FIGURE 2. Density of infected target cells

(A) $t = 0.1$ (B) $t = 0.2$ (C) $t = 0.3$
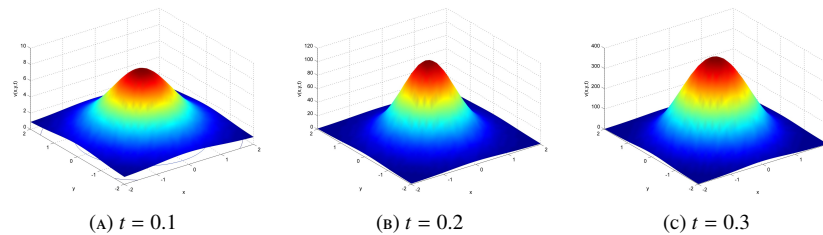
FIGURE 3. Density of free virons

5. CONCLUSION

In this paper, we suggested a more friendly approach of evaluating the Riesz-Caputo fractional differentiation matrix when solving a STFPDE using RBF methods. As shown in Table 1 above, the method is highly accurate with appropriate choice of the step sizes and can equality be applied to evaluate several fractional differential and integral operators. In addition, a two dimensional space fractional system with symmetric Riesz-Caputo fractional derivative using hybrid radial basis function pseudospectral where we applied

CSDM for evaluation differentiation matrices. In future, we will perform a comparative investigation between CSDM and Gauss–Jacobi quadrature in evaluating fractional derivatives and integrals.

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