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THE QUALITATIVE ANALYSIS OF A RATIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. The purpose of this paper is to study the dynamical behavior of positive solutions for a system of rational difference equations of the following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q},\ v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}},\ n = 0, 1, ...$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p, q, p_1, q_1$ are positive real numbers and the initial values u_{-i}, v_{-i} are non-negative real numbers for $i = 0, 1, 2$. Some examples are given to demonstrate the effectiveness of results obtained.

1. Introduction

In the last two decades, many papers appeared focusing on the investigation of the qualitative analysis of solutions of difference equations and their systems (see [1, 3, 4, 6, 7, 8, 11, 12, 16, 17, 19, 20, 21, 22]). One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in physics, computer sciences, population biology, economics, probability theory, genetics and so on. That is, the theory of difference equations gets a central position in applicable analysis. Hence, it is very valuable to study the behavior of solutions of rational difference equations (or their systems) and to discuss the local aysmptotic stability of their equilibrium points and global behavior of solutions.

In [16], Kurbanlı *et al.* studied the behavior of positive solutions of the following system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1},\ n = 0, 1, ...$$

where the initial conditions are arbitrary non-negative real numbers.

In [23], Zhang *et al.* studied the dynamical behavior of positive solutions for a system for third-order rational difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}},\ y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}},\ n = 0, 1, ...$$

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In [4], Din *et al.* studied the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}},\ y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}},\ n = 0, 1, ...$$

In [20], Touafek and Elsayed investigated the behavior of solutions of systems of difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}},\ n = 0, 1, ...$$

with a non-zero real number's initial conditions.

In [5], El-Owaidy *et al.* investigated the global behavior of the following difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p},\ n = 0, 1, ...$$

with non-negative parameters and non-negative initial values. Also, in [9, 10] the authors studied the global behavior of positive solutions for a system of difference equations of the following form

$$x_{n+1} = \frac{\alpha x_{n-1}}{1 + y_{n-2}^q},\ y_{n+1} = \frac{\beta y_{n-1}}{1 + x_{n-2}^q},\ n = 0, 1, ...$$

where the parameters α, β, q and the initial conditions are positive.

Motivated by the discussion, in this paper we investigate the equilibrium points, the local asymptotic stability of these points, the global behavior of positive solutions, the existence unbounded solutions and the existence of the prime two-periodic solutions and the rate of convergence of positive solutions of the following system

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q},\ v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}},\ n = 0, 1, ... \tag{1.1}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p, q, p_1, q_1$ are positive and the initial conditions u_{-2}, u_{-1}, u_0, v_{-2}, $v_{-1}, v_0 \in (0, \infty)$ which can be thought of as an extension of the difference equation in [5] to the system of difference equations. Thus, we say that our results extend and complement some results in the literature.

If the initial conditions $u_i = v_i$ in the system (1.1) for $i \in \{-2, -1, 0\}$ and $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \gamma_1$, $p = p_1$, $q = q_1$, then one obtain that $u_n = v_n$ for all $n \geq -2$, hence, the system (1.1) reduces to the difference equation

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma u_n^p u_{n-2}^q},\ n = 0, 1, ...$$

which was studied by Ahmed in [2]. Therefore, here we consider the case $u_i \neq v_i$ for $i \in \{-2, -1, 0\}$ and we investigate the system (1.1) basing on this condition.

It is clear that the system (1.1) can be reduced to the following system of difference equations

$$x_{n+1} = \frac{r x_{n-1}}{1 + y_n^p y_{n-2}^q},\ y_{n+1} = \frac{s y_{n-1}}{1 + x_n^{p_1} x_{n-2}^{q_1}},\ n = 0, 1, ... \tag{1.2}$$

by the change of variables $u_n = \left(\frac{\beta_1}{\gamma_1}\right)^{1/p_1+q_1} x_n$ and $v_n = \left(\frac{\beta}{\gamma}\right)^{1/p+q} y_n$ with $r = \frac{\alpha}{\beta}$ and $s = \frac{\alpha_1}{\beta_1}$. So in order to study the system (1.1), we investigate the system (1.2).

As far as we examine, there is no paper dealing with system (1.1). Therefore in this paper, we focus on system (1.1) in order to fill in the gap.

2. Preliminaries

For the completeness in the paper, we find useful to remind some basic concepts of the difference equations theory as follows:

Let us introduce the six-dimensional discrete dynamical system

$$\left.\begin{array}{l} x_{n+1}=f_1(x_n,x_{n-1},x_{n-2},y_n,y_{n-1},y_{n-2}),\\ y_{n+1}=f_2(x_n,x_{n-1},x_{n-2},y_n,y_{n-1},y_{n-2}),\end{array}\right\} \tag{2.1}$$

$n\in\mathbb{N}$ where $f_1: I_1^3\times I_2^3\to I_1$ and $f_2: I_1^3\times I_2^3\to I_2$ are continuously differentiable functions and I_1, I_2 are some intervals of real numbers.Then, for every initial conditions $(x_i,y_i)\in I_1\times I_2$, for $i=-2,-1,0$ the system (2.1) has a unique solution $\{(x_n,y_n)\}_{n=-2}^{\infty}$.

Definition 2.1. *An equilibrium point of system (2.1) is a point $(\overline{x},\overline{y})$ that satisfies*

$$\begin{array}{rcl}\overline{x} &=& f_1(\overline{x},\overline{x},\overline{x},\overline{y},\overline{y},\overline{y})\\ \overline{y} &=& f_2(\overline{x},\overline{x},\overline{x},\overline{y},\overline{y},\overline{y}).\end{array}$$

Together with the system (2.1), if we consider the associated vector map

$$F=(f_1,x_n,x_{n-1},x_{n-2},f_2,y_n,y_{n-1},y_{n-2}),$$

then the point $(\overline{x},\overline{y})$ is also called a fixed point of the vector map F.

Definition 2.2. *Let $(\overline{x},\overline{y})$ be an equilibrium point of the system (2.1).*
(i) An equilibrium point $(\overline{x},\overline{y})$ is said to be stable if for every $\varepsilon>0$, there exists $\delta>0$ such that for every initial values $(x_{-i},y_{-i})\in I_1\times I_2$, for $i\in\{0,1,2\}$ with $\sum_{i=-2}^{0}|x_i-\overline{x}|<\delta$, $\sum_{i=-2}^{0}|y_i-\overline{y}|<\delta$, implies $|x_n-\overline{x}|<\varepsilon$, $|y_n-\overline{y}|<\varepsilon$ for $n\in\mathbb{N}$.
(ii) If an equilibrium point $(\overline{x},\overline{y})$ is not stable, then it is said to be unstable.
(iii) If an equilibrium point $(\overline{x},\overline{y})$ is stable and there exists $\gamma>0$ such that

$$\sum_{i=-2}^{0}|x_i-\overline{x}|<\gamma,\ \sum_{i=-2}^{0}|y_i-\overline{y}|<\gamma$$

and $(x_n,y_n)\to(\overline{x},\overline{y})$ as $n\to\infty$, then it is said to be asymptotically stable.
(iv) If $(x_n,y_n)\to(\overline{x},\overline{y})$ as $n\to\infty$, then an equilibrium point $(\overline{x},\overline{y})$ is said to be a global attractor.
(v) If an equilibrium point $(\overline{x},\overline{y})$ is both global attractor and stable, then it is said to be globally asymptotically stable.

Definition 2.3. *If $(\overline{x},\overline{y})$ be an equilibrium point of a map*

$$F=(f_1,x_n,x_{n-1},x_{n-2},f_2,y_n,y_{n-1},y_{n-2})$$

where f_1 and f_2 are continuously differentiable functions at $(\overline{x},\overline{y})$. The linearized system of (2.1) about the equilibrium point $(\overline{x},\overline{y})$ is

$$X_{n+1}=F(X_n)=BX_n$$

where

$$X_n=\begin{pmatrix} x_n\\ x_{n-1}\\ x_{n-2}\\ y_n\\ y_{n-1}\\ y_{n-2}\end{pmatrix}$$

and B is a Jacobian matrix of the system (2.1) about the equilibrium point $(\overline{x},\overline{y})$.

Definition 2.4. *For the system $X_{n+1}=F(X_n)$, $n=0,1,...,$ of difference equations such that $\overline{X}$ is a fixed point of F. If no eigenvalues of the Jocobian matrix B about $\overline{X}$ has absolute value equal to one, then $\overline{X}$ is called hyperbolic. If there exists an eigenvalue of the Jocobian matrix B about $\overline{X}$ with absolute value equal to one, then $\overline{X}$ is called non-hyperbolic.*

The following result, known as the Linearized Stability Theorem, is very useful in determining the local stability character of the equilibrium point $(\overline{x},\overline{y})$ of system (2.1).

Theorem 2.1. *For the system $X_{n+1}=F(X_n)$, $n=0,1,...,$ of difference equations such that $\overline{X}$ is a fixed point of F. If all eigenvalues of the Jocobian matrix B about $\overline{X}$ lie inside the open unit disk $|\lambda|<1$, then $\overline{X}$ is locally asymtotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

For more details about definitions and results, we refer the reader to [13, 14, 15].

3. Main Results

In this section we will prove our main results.

Theorem 3.1. *We have the following cases for the equilibrium points of (1.2);*

- i $(\overline{x}_0,\overline{y}_0)=(0,0)$ is always the equilibrium point of system (1.2).
- ii If $r>1$ and $s>1$, then system (1.2) has the equilibrium point $(\overline{x}_1,\overline{y}_1)=((s-1)^{1/p_1+q_1},(r-1)^{1/p+q})$.
- iii If $r>1$ and $s=1$, then system (1.2) has the equilibrium point $(\overline{x}_2,\overline{y}_2)=(0,(r-1)^{1/p+q})$.
- iv if $r=1$ and $s>1$, then system (1.2) has the equilibrium point $(\overline{x}_3,\overline{y}_3)=((s-1)^{1/p_1+q_1},0)$.
- v if $r\in(0,1)$, $s=1$ and $\frac{1}{p+q}$ is an even positive integer, then system (1.2) has the equilibrium point $(\overline{x}_4,\overline{y}_4)=(0,(r-1)^{1/p+q})$.
- vi If $r=1$, $s\in(0,1)$ and $\frac{1}{p_1+q_1}$ is an even positive integer, then system (1.2) has the equilibrium point $(\overline{x}_5,\overline{y}_5)=((s-1)^{1/p_1+q_1},0)$.
- vii If $r,s\in(0,1)$ and $\frac{1}{p+q},\frac{1}{p_1+q_1}\in 2\mathbb{Z}^+$, then it also has the positive equilibrium point $(\overline{x}_6,\overline{y}_6)=((s-1)^{1/p_1+q_1},(r-1)^{1/p+q})$.

Proof. The proof is clear from the definition of equilibrium point. □

Theorem 3.2. *Assume that $\{(x_n,y_n)\}$ be a positive solution of system (1.2), one has*

$$0\leq x_n\leq\begin{cases} r^{k+1}x_{-1}, & n=2k+1\\ r^{k+1}x_0 & n=2k+2\end{cases}$$

and

$$0\leq y_n\leq\begin{cases} s^{k+1}y_{-1}, & n=2k+1\\ s^{k+1}y_0 & n=2k+2\end{cases}$$

for all $k\geq 0$.

Proof. It is clear that this assertion is true for $k = 0$. Assume that it is true for $k = k_0$. Also we have for $k = k_0 + 1$ that

$$x_n = \begin{cases} x_{2(k_0+1)+1} \le rx_{2(k_0+1)-1} = rx_{2k_0+1} \le rr^{k_0+1}x_{-1}, & n = 2(k_0+1)+1 \\ x_{2(k_0+1)+2} \le rx_{2(k_0+1)+1-1} = rx_{2k_0+2} \le rr^{k_0+1}x_0, & n = 2(k_0+1)+2 \end{cases}$$

and

$$y_n = \begin{cases} y_{2(k_0+1)+1} \le sy_{2(k_0+1)-1} = sy_{2k_0+1} \le ss^{k_0+1}y_{-1}, & n = 2(k_0+1)+1 \\ y_{2(k_0+1)+2} \le sy_{2(k_0+1)+1-1} = sy_{2k_0+2} \le ss^{k_0+1}y_0, & n = 2(k_0+1)+2 \end{cases}.$$

This completes our inductive proof. □

Corollary 3.3. *If $r < 1$ and $s < 1$, then by Theorem 3.2 $\{(x_n, y_n)\}$ converges exponentially to the equilibrium point $P_0 = (0, 0)$.*

Before we give the following theorems about the local asymptotic stability of the above equilibrium points, we build the corresponding linearized form of the system (1.2) and consider the following transformation:

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \to (f, f_1, f_2, g, g_1 g_2)$$

where $f = \frac{rx_{n-1}}{1+y_n^p y_{n-2}^q}$, $f_1 = x_n$, $f_2 = x_{n-1}$, $g = \frac{sy_{n-1}}{1+x_n^{p_1} x_{n-2}^{q_1}}$, $g_1 = y_n$, $g_2 = y_{n-1}$. The Jacobian matrix about the fixed point $(\overline{x}, \overline{y})$ under the above transformation is as follows:

$$B(\overline{x}, \overline{y}) = \begin{pmatrix} 0 & \frac{r}{1+\overline{y}^{p+q}} & 0 & -\frac{rp\overline{x}\overline{y}^{p+q-1}}{(1+\overline{y}^{p+q})^2} & 0 & -\frac{rq\overline{x}\overline{y}^{p+q-1}}{(1+\overline{y}^{p+q})^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{sp_1\overline{y}\overline{x}^{p_1+q_1-1}}{(1+\overline{x}^{p_1+q_1})^2} & 0 & -\frac{sq_1\overline{y}\overline{x}^{p_1+q_1-1}}{(1+\overline{x}^{p_1+q_1})^2} & 0 & \frac{s}{1+\overline{x}^{p_1+q_1}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where $r, s, p, q, p_1, q_1 \in (0, \infty)$.

We summarize the local asymptotic stability of the equilibria of system (1.2) as follows:

Theorem 3.4. *If $r < 1$ and $s < 1$, then the zero equilibrium point $(\overline{x}_0, \overline{y}_0)$ is locally asymptotically stable.*

Proof. The linearized system of (1.2) about the equilibrium point $(\overline{x}_0, \overline{y}_0) = (0, 0)$ is given by

$$X_{n+1} = B(\overline{x}_0, \overline{y}_0)X_n,$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$B(\overline{x}_0, \overline{y}_0) = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_0, \overline{y}_0)$ is as follows;

$$P(\lambda) = \lambda^6 - (r+s)\lambda^4 + rs\lambda^2 = 0.$$

The roots of $P(\lambda)$ are $\lambda_{1,2} = \pm\sqrt{s}$, $\lambda_{3,4} = 0$, $\lambda_{5,6} = \pm\sqrt{r}$. Since all eigenvalues of the Jacobian matrix B about $(0, 0)$ lie inside the open unit disk $|\lambda| < 1$, the zero equilibrium point $(\overline{x}_0, \overline{y}_0)$ is locally asymptotically stable. Thus, this completes the proof. □

Theorem 3.5. *(i) If $r > 1$ and $s > 1$, then the positive equilibrium point $(\overline{x}_1, \overline{y}_1)$ is locally unstable.*
(ii) If $r < 1$, $s < 1$ and $\frac{1}{p+q}, \frac{1}{p_1+q_1} \in 2\mathbb{Z}^+$, then the positive equilibrium point $(\overline{x}_6, \overline{y}_6)$ is locally unstable.

Proof. (i) The linearized system of (1.2) about the equilibrium point $(\overline{x}_1, \overline{y}_1) = ((s-1)^{1/p_1+q_1}, (r-1)^{1/p+q})$ is given by

$$X_{n+1} = B(\overline{x}_1, \overline{y}_1)X_n,$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$B(\overline{x}_1, \overline{y}_1) = \begin{pmatrix} 0 & 1 & 0 & px & 0 & qx \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ py & 0 & qy & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_1, \overline{y}_1)$ is as follows:

$$P(\lambda) = \lambda^6 + \left(-xyp^2 - 2\right)\lambda^4 + (1 - 2pqxy)\lambda^2 + \left(pqxy\left(xyp^2 + 2\right) - q^2xy - pqxy - py\left(qyp^2x^2 + qx\right)\right)$$

where

$$x = -\frac{(s-1)^{\frac{1}{p_1+q_1}}(r-1)^{\frac{p+q-1}{p+q}}}{r}$$

and

$$y = -\frac{(r-1)^{\frac{1}{p+q}}(s-1)^{\frac{p_1+q_1-1}{p_1+q_1}}}{s}.$$

It is clear that $P(\lambda)$ has a root in the interval $(1, \infty)$ since

$$\begin{aligned} P(1) &= pqxy\left(xyp^2 + 2\right) - p^2xy - q^2xy - 3pqxy - py\left(qyp^2x^2 + qx\right) \\ &= -xy\,(p+q)^2 \\ &= -\left(\frac{(s-1)^{\frac{1}{p_1+q_1}}(r-1)^{\frac{p+q-1}{p+q}}}{r}\right)\left(\frac{(r-1)^{\frac{1}{p+q}}(s-1)^{\frac{p_1+q_1-1}{p_1+q_1}}}{s}\right)(p+q)^2 \\ &= -\left(\frac{(s-1)(r-1)}{rs}\right)(p+q)^2 < 0, \text{ for } r, s > 1,\ p, q > 0 \end{aligned}$$

and

$$\lim_{\lambda \to \infty} P(\lambda) = \infty.$$

This completes the proof.

(ii) The proof is similar to the proof of (i), so it will be omitted. □

Theorem 3.6. *(i) If $r > 1$ and $s = 1$, then the equilibrium point $(\overline{x}_2, \overline{y}_2)$ is non-hyperbolic.*
(ii) If $r = 1$ and $s > 1$, then the equilibrium point $(\overline{x}_3, \overline{y}_3)$ is non-hyperbolic.
(iii) If $r < 1$, $s = 1$ and $\frac{1}{p+q}$ is an even positive integer, then the equilibrium point $(\overline{x}_4, \overline{y}_4)$ is non-hyperbolic.
(iv) If $r = 1$, $s < 1$ and $\frac{1}{p_1+q_1}$ is an even positive integer, then the equilibrium point

$(\overline{x}_5, \overline{y}_5)$ *is non-hyperbolic.*

Proof. (i) The linearized system of (1.2) about the equilibrium point $(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p+q})$ is given by

$$X_{n+1} = B(\overline{x}_2, \overline{y}_2) X_n,$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T$ and

$$B(\overline{x}_2, \overline{y}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\overline{x}_2, \overline{y}_2)$ is as follows:

$$P(\lambda) = \lambda^6 - (s+1)\lambda^4 + s\lambda^2 = 0.$$

The roots of $P(\lambda)$ are $\lambda_{1,2} = \pm\sqrt{s}$, $\lambda_{3,4} = 0$, $\lambda_{5,6} = \pm 1$. Thus, the equilibrium point $(\overline{x}_2, \overline{y}_2) = (0, (r-1)^{1/p+q})$ is non-hyperbolic point.

(ii) The proof is similar to the proof of (i), so it will be omitted.
(iii) It is clear from the proof of (i).
(iv) It is clear from the proof of (i). □

Now, we will study the global behavior of zero equilibrium point.

Theorem 3.7. *If $r < 1$ and $s < 1$, then the zero equilibrium point $(\overline{x}_0, \overline{y}_0)$ is globally asymptotically stable.*

Proof. We know by Theorem 3.4 that the equilibrium point $(\overline{x}_0, \overline{y}_0)$ of the system (1.2) is locally asymptotically stable. So, it suffices to prove for any $\{(x_n, y_n)\}_{n=-2}^{\infty}$ solution of system (1.2) that

$$\lim_{n\to\infty} (x_n, y_n) = (0, 0).$$

Since

$$0 \le x_{n+1} = \frac{r x_{n-1}}{1 + y_n^p y_{n-2}^q} < r x_{n-1}$$

and

$$0 \le y_{n+1} = \frac{s y_{n-1}}{1 + x_n^{p_1} x_{n-2}^{q_1}} < s y_{n-1},$$

we obtain by induction

$$x_{2n-1} < r^n x_{-1} \text{ and } x_{2n} < r^n x_0$$

and

$$y_{2n-1} < s^n y_{-1} \text{ and } y_{2n} < s^n y_0.$$

Thus, for $r < 1$ and $s < 1$, we obtain

$$\lim_{n\to\infty} (x_n, y_n) = (0, 0).$$

This completes the proof. □

Theorem 3.8. *Consider system (1.2) and suppose that*

$$r > 1 \text{ and } s > 1.$$

Then, we obtain the following invariant intervals, for $i \in \{0, 1, 2\}$;
i) If $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p_1+q_1}) \times ((r-1)^{1/p+q}, \infty)$, *then* $(x_n, y_n) \in (0, (s-1)^{1/p_1+q_1}) \times ((r-1)^{1/p+q}, \infty)$ *for* $n \geq 1$.
ii) If $(x_{-i}, y_{-i}) \in ((s-1)^{1/p_1+q_1}, \infty) \times (0, (r-1)^{1/p+q})$, *then* $(x_n, y_n) \in ((s-1)^{1/p_1+q_1}, \infty) \times (0, (r-1)^{1/p+q})$ *for* $n \geq 1$.

Proof. i) Let $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p_1+q_1}) \times ((r-1)^{1/p+q}, \infty)$ for $i \in \{0, 1, 2\}$. From system (1.2), we have

$$x_1 = \frac{rx_{-1}}{1 + y_0^p y_{-2}^q} < \frac{r\overline{x}_1}{1 + \overline{y}_1^{p+q}} = \overline{x}_1 = (s-1)^{1/p_1+q_1}$$

and

$$y_1 = \frac{sy_{-1}}{1 + x_0^{p_1} x_{-2}^{q_1}} > \frac{s\overline{y}_1}{1 + \overline{x}_1^{p_1+q_1}} = \overline{y}_1 = (r-1)^{1/p+q}.$$

We prove by induction that

$$(x_n, y_n) \in (0, (s-1)^{1/p_1+q_1}) \times ((r-1)^{1/p+q}, \infty), \text{ for all } n > 1. \tag{3.1}$$

Suppose that (3.1) is true for $n = k > 1$. Then, from system (1.2), we have

$$x_{k+1} = \frac{rx_{k-1}}{1 + y_k^p y_{k-2}^q} < \frac{r\overline{x}_1}{1 + \overline{y}_1^{p+q}} = \overline{x}_1 = (s-1)^{1/p_1+q_1}$$

and

$$y_{k+1} = \frac{sy_{k-1}}{1 + x_k^{p_1} x_{k-2}^{q_1}} > \frac{s\overline{y}_1}{1 + \overline{x}_1^{p_1+q_1}} = \overline{y}_1 = (r-1)^{1/p+q}.$$

Therefore, (3.1) is true for all $n \geq -2$. This completes the proof of (i). Similarly, we can obtain the proof of (ii) which will be omitted. ☐

Corollary 3.9. *Consider system (1.2) and suppose that*

$$r < 1, \ s < 1 \text{ and } \frac{1}{p+q}, \frac{1}{p_1+q_1} \in 2\mathbb{Z}^+$$

hold. Then, we obtain the following invariant intervals, for $i \in \{0, 1, 2\}$;
i) If $(x_{-i}, y_{-i}) \in (0, (s-1)^{1/p_1+q_1}) \times ((r-1)^{1/p+q}, \infty)$, *then* $(x_n, y_n) \in (0, (s-1)^{1/p_1+q_1}) \times ((r-1)^{1/p+q}, \infty)$ *for* $n \geq 1$.
ii) If $(x_{-i}, y_{-i}) \in ((s-1)^{1/p_1+q_1}, \infty) \times (0, (r-1)^{1/p+q})$, *then* $(x_n, y_n) \in ((s-1)^{1/p_1+q_1}, \infty) \times (0, (r-1)^{1/p+q})$ *for* $n \geq 1$.

Theorem 3.10. *Assume that* $r, s \in (1, \infty)$, *then there exists unbounded solutions of system (1.2).*

Proof. From Theorem 3.8, we can assume without loss of generality that the solution $\{x_n, y_n\}$ of system (1.2) is such that

$$x_n < \overline{x}_1 = (s-1)^{1/p_1+q_1} \text{ and } y_n > \overline{y}_1 = (r-1)^{1/p+q}, \text{ for } n \geq -2.$$

Then

$$x_{n+1} = \frac{rx_{n-1}}{1 + y_n^p y_{n-2}^q} < \frac{rx_{n-1}}{1 + (r-1)} = x_{n-1}$$

and

$$y_{n+1} = \frac{sy_{n-1}}{1 + x_n^{p_1} x_{n-2}^{q_1}} > \frac{sy_{n-1}}{1 + (s-1)} = y_{n-1},$$

from which it follows that

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty.$$

This completes the proof. □

Corollary 3.11. *Assume that* $r, s \in (0,1)$ *and* $\frac{1}{p+q}, \frac{1}{p_1+q_1} \in 2\mathbb{Z}^+$*, then there exists unbounded solutions of system (1.2).*

Theorem 3.12. *If* $r = s = 1$*, then system (1.2) possesses the prime period two solution which is one of the following forms;*

$$\begin{array}{c} \ldots,(0,y),(0,w),(0,y),(0,w),\ldots \\ \ldots,(0,y),(x,0),(0,y),(x,0),\ldots \\ \ldots,(x,0),(z,0),(x,0),(z,0),\ldots \\ \ldots,(x,0),(0,w),(x,0),(0,w),\ldots \end{array}$$

with $x, y, z, w > 0$*.*

Proof. Suppose that

$$\ldots,(x,y),(z,w),(x,y),(z,w),\ldots$$

be a prime period-two solution of system (1.2). Then we have

$$x = \frac{rx}{1+w^{p+q}},\ y = \frac{ry}{1+z^{p+q}} \tag{3.2}$$

and

$$z = \frac{sz}{1+y^{p_1+q_1}},\ w = \frac{sw}{1+x^{p_1+q_1}} \tag{3.3}$$

such that $x \neq y$ and $z \neq w$. Firstly, we consider the case both $x \neq 0$ and $y \neq 0$. Then, we obtain from (3.2) that $z = w = (r-1)^{1/p+q}$ which is a contradiction. Similarly, the case both $z \neq 0$ and $w \neq 0$ is impossible with (3.3). Thus, one of them must be equal to zero. Then, we can assume that $x = 0$, $y \neq 0$, $z = 0$, $w \neq 0$. In this sense, we obtain from (3.2) and (3.3) that $r = s = 1$. Therefore,

$$\ldots,(0,y),(0,w),(0,y),(0,w),\ldots$$

is a prime two periodic solution of system (1.2) with $y, w > 0$. The other cases are similar and will be omitted.

In contrast, if $r = s = 1$ and choose the initial conditions such as $x_{-2} = x_0 = y_{-2} = y_0 = 0$, $x_{-1} = \delta$ and $y_{-1} = \delta_1$, then we can see by induction that

$$(\delta,0),(\delta_1,0),(\delta,0),(\delta_1,0),\ldots$$

is the prime period two solution of system (1.2). The other cases are similar and will be omitted. Thus, the proof is completed. □

4. Rate of Convergence

In this section, we will study the rate of convergence of a solution that converges to the equilibrium point $(0,0)$ of the system (1.2). The following result gives the rate of convergence of the solution of a system of difference equations:

$$X_{n+1} = [A + B(n)]X_n \tag{4.1}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0, \text{ when } n \to \infty, \tag{4.2}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm.

Proposition 4.1. [18] *Assume that condition (4.2) holds, if X_n is a solution of (4.1), then either $X_n = 0$ for all large n or*

$$\theta = \lim_{n\to\infty} \sqrt[n]{\|X_n\|}$$

or

$$\theta = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A.

Assume that $\lim_{n\to\infty} x_n = \overline{x}$ and $\lim_{n\to\infty} y_n = \overline{y}$, we will find a system of limitting equations for the system (1.2). The error terms are given as

$$\begin{aligned} x_{n+1} - \overline{x} &= \sum_{i=0}^{2} A_i(x_{n-i} - \overline{x}) + \sum_{i=0}^{2} B_i(y_{n-i} - \overline{y}) \\ y_{n+1} - \overline{y} &= \sum_{i=0}^{2} C_i(x_{n-i} - \overline{x}) + \sum_{i=0}^{2} D_i(y_{n-i} - \overline{y}). \end{aligned}$$

Set $e_n^1 = x_n - \overline{x}$, $e_n^2 = y_n - \overline{y}$; therefore, it follows that

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^{2} A_i e_{n-i}^1 + \sum_{i=0}^{2} B_i e_{n-i}^2 \\ e_{n+1}^2 &= \sum_{i=0}^{2} C_i e_{n-i}^1 + \sum_{i=0}^{2} D_i e_{n-i}^2 \end{aligned}$$

where

$$\begin{aligned} A_0 &= 0,\ A_1 = \frac{r}{1 + y_n^p y_{n-2}^q},\ A_2 = 0, \\ B_0 &= -\frac{rpx_{n-1}y_n^{p-1}y_{n-2}^q}{(1 + y_n^p y_{n-2}^q)^2},\ B_1 = 0,\ B_2 = -\frac{rqx_{n-1}y_n^p y_{n-2}^{q-1}}{(1 + y_n^p y_{n-2}^q)^2}, \\ C_0 &= -\frac{sp_1 y_{n-1}x_n^{p_1-1}x_{n-2}^{q_1}}{(1 + x_n^{p_1} x_{n-2}^{q_1})^2},\ C_1 = 0,\ C_2 = -\frac{sq_1 y_{n-1}x_n^{p_1}x_{n-2}^{q_1-1}}{(1 + x_n^{p_1} x_{n-2}^{q_1})^2}, \\ D_0 &= 0,\ D_1 = \frac{s}{1 + x_n^{p_1} x_{n-2}^{q_1}},\ D_2 = 0. \end{aligned}$$

If we consider the limitting case, it is clear that

$$\begin{aligned}
\lim_{n\to\infty} A_0 &= 0,\ \lim_{n\to\infty} A_1 = \frac{r}{1+\overline{y}^{p+q}} \text{ and } \lim_{n\to\infty} A_2 = 0 \\
\lim_{n\to\infty} B_0 &= -\frac{rp\overline{x}\overline{y}^{p+q-1}}{(1+\overline{y}^{p+q})^2},\ \lim_{n\to\infty} B_1 = 0 \text{ and } \lim_{n\to\infty} B_2 = -\frac{rq\overline{x}\overline{y}^{p+q-1}}{(1+\overline{y}^{p+q})^2}, \\
\lim_{n\to\infty} C_0 &= -\frac{sp_1\overline{y}\overline{x}^{p_1+q_1-1}}{(1+\overline{x}^{p_1+q_1})^2},\ \lim_{n\to\infty} C_1 = 0 \text{ and } \lim_{n\to\infty} C_2 = -\frac{sq_1\overline{y}\overline{x}^{p_1+q_1-1}}{(1+\overline{x}^{p_1+q_1})^2}, \\
\lim_{n\to\infty} D_0 &= 0,\ \lim_{n\to\infty} D_1 = \frac{s}{1+\overline{x}^{p_1+q_1}} \text{ and } \lim_{n\to\infty} D_2 = 0
\end{aligned}$$

Thus, the limitting system of error terms at $(0,0)$ can be written as follows:

$$E_{n+1} = KE_n,$$

where

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}_{6\times 1}$$

and

$$K = J_F(0,0) = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{6\times 6}.$$

Using Theorem 4.1, we have the following result.

Theorem 4.2. *Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a solution of the system (1.2) such that*

$$\lim_{n\to\infty} x_n = \overline{x} \text{ and } \lim_{n\to\infty} y_n = \overline{y}$$

where $(\overline{x}, \overline{y}) = (0,0)$. Then, the error vector E_n of every solution of the system (1.2) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|} = |\lambda J_F(0,0)|\,,\ \ \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda J_F(0,0)|$$

where $\lambda J_F(0,0)$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point $(0,0)$.

5. Numerical Examples

In order to verify our theoretical results we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions of the system (1.2). All plots in this section are drawn with Mathematica.

Example (1) Consider the system (1.2) with the initial values $x_{-2} = 0.6$, $x_{-1} = 2$, $x_0 = 0.1$, $y_{-2} = 0.2$, $y_{-1} = 1$, $y_0 = 0.8$. Also, if we choose the parameters as $r = 0.5$, $s = 0.1$, $p = 2$, $q = 5$, $p_1 = 4$, $q_1 = 3$, then we obtain the following system

$$x_{n+1} = \frac{0.5x_{n-1}}{1 + y_n^2 y_{n-2}^5},\ y_{n+1} = \frac{0.1y_{n-1}}{1 + x_n^4 x_{n-2}^3},\ n = 0, 1, ... \quad (5.1)$$

The plot of system (5.1) is shown in Figure 1.

Example (2) Consider the system (1.2) with the initial values $x_{-2} = 0.6$, $x_{-1} = 2$, $x_0 = 0.1$, $y_{-2} = 0.2$, $y_{-1} = 1$, $y_0 = 0.8$. Also, if we choose the parameters as $r = 0.5$, $s = 1.1$, $p = 2$, $q = 5$, $p_1 = 4$, $q_1 = 3$, then we obtain the following system

$$x_{n+1} = \frac{0.5x_{n-1}}{1 + y_n^2 y_{n-2}^5},\ y_{n+1} = \frac{1.1y_{n-1}}{1 + x_n^4 x_{n-2}^3},\ n = 0, 1, ... \quad (5.2)$$

The plot of system (5.2) is shown in Figure 2.

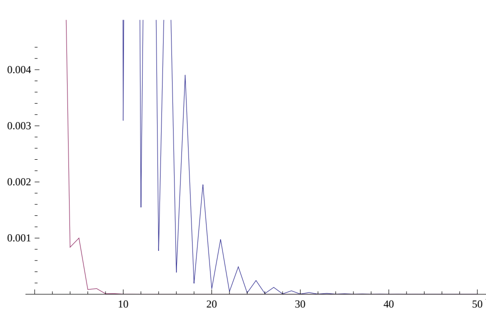

Figure 1. The plot of system (5.1)

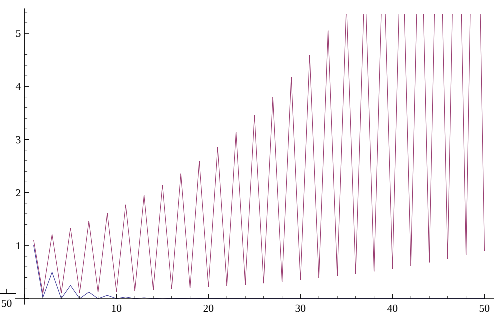

Figure 2. The plot of system (5.2)

Example (3) Consider the system (1.2) with the initial values $x_{-2} = 0.6$, $x_{-1} = 2$, $x_0 = 0.1$, $y_{-2} = 0.2$, $y_{-1} = 1$, $y_0 = 0.8$. Also, if we choose the parameters as $r = 1.2$, $s = 1.1$, $p = 2$, $q = 5$, $p_1 = 4$, $q_1 = 3$, then we obtain the following system

$$x_{n+1} = \frac{1.2x_{n-1}}{1 + y_n^2 y_{n-2}^5},\ y_{n+1} = \frac{1.1y_{n-1}}{1 + x_n^4 x_{n-2}^3},\ n = 0, 1, ... \quad (5.3)$$

The plot of system (5.3) is shown in Figure 3.

Example (4) Consider the system (1.2) with the initial values $x_{-2} = 0.6$, $x_{-1} = 2$, $x_0 = 0.1$, $y_{-2} = 0.2$, $y_{-1} = 1$, $y_0 = 0.8$. Also, if we choose the parameters as $r = 1$, $s = 1$, $p = 2$, $q = 5$, $p_1 = 4$, $q_1 = 3$, then we obtain the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + y_n^2 y_{n-2}^5},\ y_{n+1} = \frac{y_{n-1}}{1 + x_n^4 x_{n-2}^3},\ n = 0, 1, ... \quad (5.4)$$

The plot of system (5.4) is shown in Figure 4.

6. Conclusions

This paper is a natural extension of the articles [2, 5, 9, 10]. We have studied some dynamics of a six dimensional discrete system. We have investigated the steady states of the system (1.2) in detail. Also, we have studied the stability character of these points using the linearization method . The main aim of dynamical systems theory is to approach the global behavior and the rate of convergence. So, here we have studied the global asymptotic stability and the rate of convergence of the

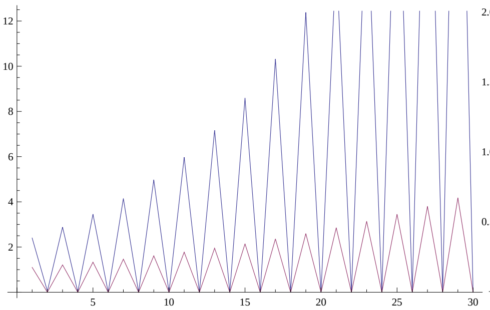

Figure 3. The plot of system (5.3)

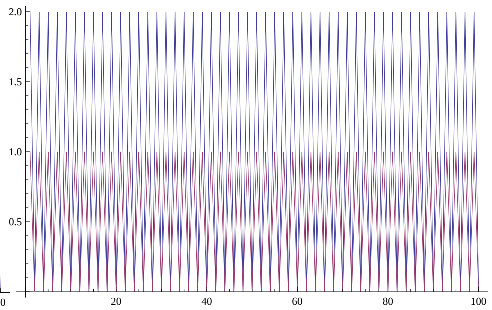

Figure 4. The plot of system (5.4)

zero equilibrium point of the system. Also, the existence unbounded solutions and the periodic nature of positive solutions of this system are investigated. Even if it will be possible to obtain analytical conditions, it would be quite difficult to deal with them. So, numerical simulations have been used to verify the correctness of analytical results.

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