

APPROXIMATE SOLUTIONS TO INITIAL VALUE PROBLEM FOR DIFFERENTIAL EQUATION OF VARIABLE ORDER

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ABSTRACT. Some experts claim that the Riemann-Liouville variable order fractional integral didn't have semigroup property. We give examples to support and validate this judgment. This property brought us extreme difficulty when we consider the existence of solutions of variable order fractional differential equations. In this work, based on the definitions of a partition of a finite interval and a piecewise constant function with respect to this partition, we introduce the concept of approximate solution to an initial value problem for differential equations of variable order. And then, by our discussion and analysis, we investigate the existence of approximate solutions to the initial value problem for differential equations of variable order.

1. VARIABLE ORDER INTEGRALS AND DERIVATIVES

In this paper, we observe and study the existence of the solutions to the following variable order fractional differential equation

$$D_{0+}^{p(t)} x(t) = f(t, x), x(0) = 0, 0 < t < +\infty, \quad (1.1)$$

where $0 < p(t) < 1$, $f(t, x)$ are given real functions, and $D_{0+}^{p(t)}$ denotes derivative of variable order defined by

$$D_{0+}^{p(t)} x(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} x(s) ds, t > 0. \quad (1.2)$$

$$I_{0+}^{1-p(t)} x(t) = \int_0^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} x(s) ds, t > 0, \quad (1.3)$$

denotes integral of variable order $1 - p(t)$, for details, please see [1]

The operators of variable order, which fall into a more complex category, are the derivatives and integrals whose orders are the functions of certain variables. Recently, operators and differential equations of variable order have been considered, see [1]-[3], [5]-[16].

We notice that, if the order $p(t)$ is a constant function q , then the Riemann-Liouville variable order fractional derivative (1.2) and the variable order fractional

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integral (1.3) are the Riemann-Liouville fractional derivative and integral, respectively. (see [2], [4]). We know there are some important properties as following (Let $-\infty < b < \infty$)

Lemma 1.1. [4] The Riemann-Liouville fractional integral defined for function $x(t) \in L(0, b)$ exists almost everywhere.

Lemma 1.2. [4] The equality $I_{0+}^{\gamma} I_{0+}^{\delta} f(t) = I_{0+}^{\delta} I_{0+}^{\gamma} f(t) = I_{0+}^{\gamma+\delta} f(t)$, $0 < \gamma < 1, 0 < \delta < 1$ holds for $f \in L(0, b)$.

The Lemma 1.2 is semigroup property for Riemann-Liouville fractional integral, which is very critical in obtaining the following Lemmas 1.3 – 1.5. In other words, without Lemma 1.2, ones couldn't have Lemmas 1.3 – 1.5, for details, please see [4].

Lemma 1.3. [4] The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $0 < \gamma < 1$ holds for $f \in L(0, b)$.

Lemma 1.4. [4] Let $0 < \alpha < 1$, then the differential equation

$$D_{0+}^{\alpha} u = 0$$

has solution

$$u(t) = ct^{\alpha-1}, c \in R.$$

Lemma 1.5. [4] Let $0 < \alpha < 1$, $u \in L(0, b)$, $D_{0+}^{\alpha} u \in L(0, b)$. Then the following equality holds

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + ct^{\alpha-1}, c \in R.$$

These properties play a very important role in considering the existence of the solutions of differential equations for the Riemann-Liouville fractional derivative, for details, please see [4], [17]–[22]. However, from [2], [3], for general function $p(t), q(t)$, we notice that the semigroup property don't hold, i.e., $I_{a+}^{p(t)} I_{a+}^{q(t)} \neq I_{a+}^{p(t)+q(t)}$. Based on this fact, whether the variable order fractional integration operators have these properties like Lemmas 1.2 – 1.5 is not sure. Without these properties for variable order derivative and integral, we can hardly consider the existence of solutions of differential equations for variable order derivative by means of nonlinear functional analysis (for instance, some fixed point theorems). In this paper, based on the definitions of a partition of a finite interval and a piecewise constant function with respect to this partition, we introduce the concept of approximate solutions to the problem (1.1). And then, according to our discussion and analysis, we explore the existence of the approximate solutions of the problem (1.1).

This paper is organized as follows. In section 2, we provide some facts to the variable order integral and derivative through several examples. Also, we state some results which will play a very important role in obtaining our main results. In section 3, we set forth our main existence result of the approximate solutions for the initial value problem (1.1). Finally, an example is given.

2. SOME PRELIMINARIES ON APPROXIMATE SOLUTIONS

In this section, we give some preliminaries on approximate solutions to the initial value problem (1.1). First of all, we use an example to illustrate the claim in [2], [3]: for general function $p(t), q(t)$, the Riemann-Liouville variable order fractional integral doesn't have the semigroup property.

Example 2.1. Let $p(t) = \frac{t}{6} + \frac{1}{3}$, $q(t) = \frac{t}{4} + \frac{1}{4}$, $f(t) = 1, 0 \leq t \leq 3$. Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=1}$.

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) &= \int_0^t \frac{(t-s)^{\frac{t}{6}+\frac{1}{3}-1}}{\Gamma(\frac{t}{6}+\frac{1}{3})} \int_0^s \frac{(s-\tau)^{\frac{s}{4}+\frac{1}{4}-1}}{\Gamma(\frac{s}{4}+\frac{1}{4})} d\tau ds \\ &= \int_0^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds \\ &= \int_0^1 \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds + \int_1^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds. \end{aligned}$$

We set $M_1 = \max_{1 \leq t \leq 3} |\frac{1}{\Gamma(p(t))}|$ and $M_2 = \max_{1 \leq s \leq 3} |\frac{1}{\Gamma(\frac{5}{4}+\frac{s}{4})}|$. For $1 \leq t \leq 3$, it holds

$$\begin{aligned} \left| \int_1^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds \right| &= \left| \int_1^t 3^{\frac{t}{6}-\frac{2}{3}} \left(\frac{t-s}{3}\right)^{\frac{t}{6}-\frac{2}{3}} \frac{s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds \right| \\ &\leq M_1 M_2 \int_1^t 3^{\frac{1}{2}-\frac{2}{3}} \left(\frac{t-s}{3}\right)^{\frac{1}{6}-\frac{2}{3}} s ds \\ &\leq M_1 M_2 \int_1^t 3^{\frac{1}{3}} (t-s)^{-\frac{1}{2}} t ds \\ &= 2 \times 3^{\frac{4}{3}} M_1 M_2 (t-1)^{\frac{1}{2}}, \end{aligned}$$

hence, we have

$$\left[\int_1^t \frac{(t-s)^{\frac{t}{6}-\frac{2}{3}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{t}{6}+\frac{1}{3})\Gamma(\frac{5}{4}+\frac{s}{4})} ds \right]_{t=1} = 0.$$

So, we get

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{-\frac{1}{2}} s^{\frac{1}{4}+\frac{s}{4}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{s}{4})} ds \approx 0.458$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{p(1)+q(1)-1}}{\Gamma(p(1)+q(1))} ds = \frac{1}{\Gamma(1+\frac{1}{2}+\frac{1}{2})} = 1.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=1}.$$

Without semigroup property of the Riemann-Liouville variable order fractional integral, we could assure that the variable order fractional integration operator of non-constant continuous functions $p(t)$ for $x(t)$ doesn't have the properties like Lemmas 1.3-1.5. Consequently, we couldn't transform differential equations of variable order into an integral equation.

Let $L[x(t); s]$, $L[I_{0+}^{p(t)} x(t); s]$, $L[D_{0+}^{p(t)} x(t); s]$ denote the Laplace transforms of functions $x(t)$, $I_{0+}^{p(t)} x(t)$ and $D_{0+}^{p(t)} x(t)$. We find out that there is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t)} x(t); s]$, as a result, there is also no explicit connection between $L[x(t); s]$ and $L[D_{0+}^{p(t)} x(t); s]$.

Example 2.2. Let $p(t) = \frac{1}{\sqrt{t+1}}, t \geq 0$. We consider the Laplace transforms of functions $t^2(t \geq 0)$ and $I_{0+}^{p(t)}t^2(t \geq 0)$. We could know that

$$L[t^2; s] = \int_0^\infty e^{-st}t^2 dt = \frac{3}{s^3}, \quad (2.1)$$

$$\begin{aligned} L[I_{0+}^{(t+1)^{-\frac{1}{2}}}t^2; s] &= \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{(t+1)^{-\frac{1}{2}}-1}}{\Gamma((t+1)^{-\frac{1}{2}})} \tau^2 d\tau dt \\ &= \int_0^\infty e^{-st} \int_\tau^\infty \frac{(t-\tau)^{(t+1)^{-\frac{1}{2}}-1}}{\Gamma((t+1)^{-\frac{1}{2}})} \tau^2 dt d\tau \\ &= \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{r^{(\tau+r+1)^{-\frac{1}{2}}-1}}{\Gamma((\tau+r+1)^{-\frac{1}{2}})} \tau^2 dr d\tau \\ &= \int_0^\infty e^{-s\tau} \tau^2 \int_0^\infty e^{-sr} \frac{r^{(\tau+r+1)^{-\frac{1}{2}}-1}}{\Gamma((\tau+r+1)^{-\frac{1}{2}})} dr d\tau \\ &= \int_0^\infty e^{-s\tau} \tau^2 L\left[\frac{r^{(\tau+r+1)^{-\frac{1}{2}}-1}}{\Gamma((\tau+r+1)^{-\frac{1}{2}})}; s\right] d\tau. \end{aligned} \quad (2.2)$$

By (2.1) or (2.2), we guarantee there is no explicit connection between $L[t^2; s]$ and $L[I_{0+}^{(t+1)^{-\frac{1}{2}}}t^2; s]$.

In view of this example and the connection between the Laplace transforms of function $x(t)$ and its derivative $x'(t)$, we couldn't obtain the Laplace transform formula for variable order fractional derivative (1.2). Based on these facts, we couldn't get the explicit expression of the solutions for the problem (1.1).

The following result is needed in our next analysis of main result.

Lemma 2.3. If $p : [0, b] \rightarrow (0, 1)$ ($0 < b < +\infty$) is a real continuous function. Then for $h \in C_r[0, b] \doteq \{h(t) \in C(0, b], t^r h(t) \in C[0, b]\}$ ($0 \leq t \leq \min_{0 \leq t \leq b} |p(t)|$), then the variable order fractional integral $I_{0+}^{p(t)}h(t)$ exists for any points on $[0, b]$.

Proof. According to the continuity of function $\Gamma(p(t))$, we can claim that $L_p = \max_{0 \leq t \leq b} |\frac{1}{\Gamma(p(t))}|$ exists. We let $p_* = \min_{0 \leq t \leq b} |p(t)|$. Thus, for $0 \leq s \leq t \leq b$, we have

$$\text{if } 0 \leq b \leq 1, \text{ then } b^{p(s)-1} \leq b^{p_*-1};$$

$$\text{if } 1 < b < +\infty, \text{ then } b^{p(s)-1} < 1.$$

Thus, for $-\infty < b < +\infty$, we know

$$b^{p(s)-1} \leq \max\{1, b^{p_*-1}\} \doteq b^*.$$

For $h \in C_r[0, b]$, by the definition of the variable order fractional integral (1.3), we have

$$\begin{aligned} |I_{0+}^{p(t)}h(t)| &\leq L_P \int_0^t (t-s)^{p(s)-1} |h(s)| ds \\ &= L_P \int_0^t b^{p(s)-1} \left(\frac{t-s}{b}\right)^{p(s)-1} s^{-r} s^r |h(s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq L_P b^* \int_0^t \left(\frac{t-s}{b}\right)^{p_*-1} s^{-r} \max_{0 \leq s \leq b} s^r |h(s)| ds \\
&\leq L_P \max_{0 \leq s \leq b} s^r |h(s)| b^* b^{1-p_*} \int_0^t (t-s)^{p_*-1} s^{-r} ds \\
&\leq \frac{L_P \max_{0 \leq s \leq b} s^r |h(s)| b^* b^{1-p_*} \Gamma(p_*) \Gamma(1-r)}{\Gamma(1-r+p_*)} b^{p_*-r} < \infty,
\end{aligned}$$

which implies that the variable order fractional integral $I_{0+}^{p(t)} h(t)$ exists for any points on $[0, b]$. \square

Lemma 2.4. Let $p : [0, b] \rightarrow (0, 1)$ be a continuous function, then $I_{0+}^{p(t)} x(t) \in C[0, b]$ for $x \in C[0, b]$.

Proof. For $x \in C[0, b]$, $t, t_0 \in [0, b]$, without loss of generalized, let $t > t_0$. Thus, we obtain

$$\begin{aligned}
&|I_{0+}^{p(t)} x(t) - I_{0+}^{p(t_0)} x(t_0)| \\
&= \left| \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} x(s) ds - \int_0^{t_0} \frac{(t_0-s)^{p(t_0)-1}}{\Gamma(p(t_0))} x(s) ds \right| \\
&= \left| \int_0^1 \frac{t^{p(t)}(1-r)^{p(t)-1}}{\Gamma(p(t))} x(tr) dr - \int_0^1 \frac{t_0^{p(t_0)}(1-r)^{p(t_0)-1}}{\Gamma(p(t_0))} x(t_0 r) dr \right| \\
&= \left| \int_0^1 \left[\frac{t^{p(t)}(1-r)^{p(t)-1} x(tr)}{\Gamma(p(t))} - \frac{t_0^{p(t)}(1-r)^{p(t)-1} x(tr)}{\Gamma(p(t))} \right] dr \right. \\
&\quad + \int_0^1 \left[\frac{t_0^{p(t)}(1-r)^{p(t)-1} x(tr)}{\Gamma(p(t))} - \frac{t_0^{p(t)}(1-r)^{p(t_0)-1} x(tr)}{\Gamma(p(t))} \right] dr \\
&\quad + \int_0^1 \left[\frac{t_0^{p(t)}(1-r)^{p(t_0)-1} x(tr)}{\Gamma(p(t))} - \frac{t_0^{p(t)}(1-r)^{p(t_0)-1} x(tr)}{\Gamma(p(t_0))} \right] dr \\
&\quad + \int_0^1 \left[\frac{t_0^{p(t_0)}(1-r)^{p(t_0)-1} x(tr)}{\Gamma(p(t_0))} - \frac{t_0^{p(t_0)}(1-r)^{p(t_0)-1} x(tr)}{\Gamma(p(t_0))} \right] dr \\
&\quad \left. + \int_0^1 \left[\frac{t_0^{p(t_0)}(1-r)^{p(t_0)-1} x(tr)}{\Gamma(p(t_0))} - \frac{t_0^{p(t_0)}(1-r)^{p(t_0)-1} x(t_0 r)}{\Gamma(p(t_0))} \right] dr \right| \\
&\leq M_x \int_0^1 \frac{(1-r)^{p(t)-1}}{|\Gamma(p(t))|} |t^{p(t)} - t_0^{p(t)}| dr + M_x \int_0^1 \frac{(1-r)^{p(t_0)-1}}{|\Gamma(p(t_0))|} |t_0^{p(t_0)} - t_0^{p(t)}| dr \\
&\quad + M_x \int_0^1 \frac{t_0^{p(t)}}{|\Gamma(p(t))|} |(1-r)^{p(t)-1} - (1-r)^{p(t_0)-1}| dr \\
&\quad + \int_0^1 \frac{t_0^{p(t_0)}(1-r)^{p(t_0)}}{|\Gamma(p(t_0))|} |x(tr) - x(t_0 r)| dr \\
&\quad + M_x \int_0^1 t_0^{p(t)}(1-r)^{p(t_0)-1} \left| \frac{1}{\Gamma(p(t))} - \frac{1}{\Gamma(p(t_0))} \right| dr \\
&\doteq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

(here, $M_x = \max_{0 \leq t \leq b} |x(t)|$) it follows from the continuity of functions $t^{p(t)}$, $x(t)$, $\frac{1}{\Gamma(p(t))}$ and $t_0^{p(t)}$ that I_1, I_3, I_4 and I_5 are continuous at point t_0 . Obviously, according to $\int_0^1 (1-r)^{p(t)-1} dr = \frac{1}{p(t)}$, we get that I_2 is continuous at point t_0 . As a result, $I_{0+}^{p(t)} x(t)$ is continuous at t_0 , by the arbitrary of t_0 , we get that $I_{0+}^{p(t)} x(t) \in C[0, b]$ for $x \in C[0, b]$. The proof is completed. \square

Throughout this paper, we assume that

(A₁) Let $p : [0, +\infty) \rightarrow (0, 1)$ be a continuous function satisfying

$$\lim_{t \rightarrow +\infty} p(t) = \rho, 0 \leq \rho < +\infty. \quad (2.3)$$

Lemma 2.5. Let condition (A₁) hold. Then there exist positive constant T and intervals $[0, T_1], (T_1, T_2], \dots, (T_{n^*-1}, T], (T, +\infty)$, and function $q : [0, +\infty) \rightarrow (0, 1)$ defined by

$$q(t) = \sum_{k=0}^{n^*-1} p_k I_k(t) + \rho I_\rho(t) \quad t \in [0, +\infty), \quad (2.4)$$

where $I_k(t)$ is the indicator of the interval $[T_k, T_{k+1}]$ ($k = 0, 1, 2, \dots, n^* - 1$, here $T_0 = 0, T_{n^*} = T$), i.e. $I_k(t) = 1$ for $t \in [T_k, T_{k+1}]$, $I_k(t) = 0$ for t lying in elsewhere; $I_\rho(t)$ is the indicator of interval $(T, +\infty)$, i.e. $I_\rho(t) = 1$ for $t \in (T, +\infty)$, $I_\rho(t) = 0$ for t lying in elsewhere, such that for arbitrary small $\varepsilon > 0$,

$$|p(t) - q(t)| < \varepsilon, \quad 0 \leq t < +\infty. \quad (2.5)$$

holds.

Proof. By (2.3), for $\forall \varepsilon > 0$, there exists $T > 0$, such that

$$|p(t) - \rho| < \varepsilon, \quad t > T. \quad (2.6)$$

We know that $p : [0, T] \rightarrow (0, 1)$ is a continuous function. Since $p(t)$ is right continuous at point 0, then, for arbitrary small $\varepsilon > 0$, there has $\delta_0 > 0$ such that

$$|p(t) - p(0)| < \varepsilon, \quad \text{for } 0 \leq t \leq \delta_0. \quad (2.7)$$

We take point $\delta_0 \doteq T_1$ (if $T_1 \leq T$, we consider continuity of $p(t)$ at point T_1 , otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_1 , so, for arbitrary small $\varepsilon > 0$, there is $\delta_1 > 0$ such that

$$|p(t) - p(T_1)| < \varepsilon, \quad \text{for } T_1 \leq t \leq T_1 + \delta_1, \quad (2.8)$$

We take point $T_1 + \delta_1 \doteq T_2$ (if $T_2 \leq T$, we consider continuity of $p(t)$ at point T_2 , otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_2 , so, for arbitrary small $\varepsilon > 0$, there is $\delta_2 > 0$ such that

$$|p(t) - p(T_2)| < \varepsilon, \quad \text{for } T_2 \leq t \leq T_2 + \delta_2. \quad (2.9)$$

We take point $T_2 + \delta_2 \doteq T_3$ (if $T_3 \leq T$, we consider continuity of $p(t)$ at point T_3 , otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_3 , so, for arbitrary small $\varepsilon > 0$, there is $\delta_3 > 0$ such that

$$|p(t) - p(T_3)| < \varepsilon, \quad \text{for } T_3 \leq t \leq T_3 + \delta_3. \quad (2.10)$$

Continuous this analysis procedure, we could obtain there exists $\delta_{n^*-2} > 0, \delta_{n^*-1} > 0$ ($n^* \in N$) such that $T_{n^*-2} + \delta_{n^*-2} \doteq T_{n^*-1} < T, T_{n^*-1} + \delta_{n^*-1} \geq T$, and that

$$|p(t) - p(T_{n^*-1})| < \varepsilon, \quad \text{for } T_{n^*-1} \leq t \leq T_{n^*-1} + \delta_{n^*-1}, \quad (2.11)$$

From (2.6) – (2.11), we could let

$$p(0) \doteq p_0, p(T_1) \doteq p_1, p(T_2) \doteq p_2, p(T_3) \doteq p_3, \dots, p(T_{n^*-1}) \doteq p_{n^*-1}.$$

Hence, from the previous arguments, it holds

$$\left\{ \begin{array}{l} |p(t) - p_0| < \varepsilon, \text{ for } t \in [0, T_1], \\ |p(t) - p_1| < \varepsilon, \text{ for } t \in (T_1, T_2], \\ \vdots \\ |p(t) - p_{n^*-1}| < \varepsilon, \text{ for } t \in (T_{n^*-1}, T], \\ |p(t) - \rho| < \varepsilon, \text{ for } t \in (T, +\infty). \end{array} \right. \quad (2.12)$$

Thus, we complete this proof. □

Remark 2.6. $P = \{[0, T_1], (T_1, T_2], \dots, (T_{n^*-1}, T]\}$ is a partition of interval $[0, T]$, $\omega(t) = \sum_{k=0}^{n^*-1} p_k I_k(t)$ is a piecewise constant function with respect to P .

Lemma 2.7. Let condition (A_1) hold. Then, for arbitrary small $\varepsilon > 0$, function $q(t)$ defined in (2.4) and

$$x(t) = \left\{ \begin{array}{l} x_0(t) \in C[0, T_1], \\ x_1(t) \in C[0, T_2], \\ \vdots \\ x_{n^*-2}(t) \in C[0, T_{n^*-1}], \\ x_{n^*-1}(t) \in C[0, T], \\ x_T(t) \in C[0, +\infty), \end{array} \right.$$

we have

$$\left\{ \begin{array}{l} |I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-q(t)} x(t)| < \varepsilon, 0 \leq t \leq T_i, i = 1, 2, \dots, n^*, (T_{n^*} = T), \\ |I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-q(t)} x(t)| < \varepsilon, 0 \leq t < +\infty. \end{array} \right. \quad (2.13)$$

Proof. It follows from the continuity of $p(t)$ on closed interval $[0, T]$ that there exists $t^* \in [0, T]$ such that

$$p(t^*) = \max_{0 \leq t \leq T} p(t).$$

As well, from the continuity of $a^t, a \geq 0, t > 0, \Gamma(1 - p(t))$ and (2.12), we have

$$|(t - s)^{p(t^*)-p(t)} - (t - s)^{p(t^*)-p_i}| < \varepsilon, \quad 0 \leq s \leq t < +\infty, \quad (2.14)$$

$$\left| \frac{1}{\Gamma(1 - p(t))} - \frac{1}{\Gamma(1 - p_i)} \right| < \varepsilon, \quad 0 \leq s \leq t < +\infty, \quad (2.15)$$

$i = 0, 1, 2, \dots, n^* - 1, n^*, p_{n^*} = \rho$.

By the definitions of $x(t)$ and $q(t)$, for $t \in [0, T_1]$, one has

$$\left| I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-q(t)} x(t) \right|$$

$$\begin{aligned}
&= \left| \int_0^t \left[\frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_0}}{\Gamma(1-p_0)} \right] x_0(s) ds \right| \\
&\leq \int_0^t \left| \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_0}}{\Gamma(1-p_0)} \right| \cdot \max_{0 \leq s \leq T_1} |x_0(s)| ds \\
&\quad + \int_0^t \left| \frac{(t-s)^{-p_0}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_0}}{\Gamma(1-p_0)} \right| \cdot \max_{0 \leq s \leq T_1} |x_0(s)| ds \\
&= \int_0^t \left| \frac{(t-s)^{-p(t^*)}}{\Gamma(1-p(t))} \left[(t-s)^{p(t^*)-p(t)} - (t-s)^{p(t^*)-p_0} \right] \right| \cdot \max_{0 \leq s \leq T_1} |x_0(s)| ds \\
&\quad + \int_0^t (t-s)^{-p_0} \left| \frac{1}{\Gamma(1-p(t))} - \frac{1}{\Gamma(1-p_0)} \right| \cdot \max_{0 \leq s \leq T_1} |x_0(s)| ds.
\end{aligned}$$

According to (2.14), (2.15), we get that (2.13) holds for $t \in [0, T_1]$.

By the definitions of $x(t)$ and $q(t)$, for $t \in [0, T_2]$, it holds that

$$\begin{aligned}
&|I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-q(t)} x(t)| \\
&\leq \int_0^t \left| \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t))} \right| \cdot |x(s)| ds \\
&\leq \int_0^t \left| \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-q(t)}}{\Gamma(1-p(t))} \right| \cdot |x(s)| ds + \int_0^t \left| \frac{(t-s)^{-q(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t))} \right| \cdot |x(s)| ds \\
&= \int_0^{T_1} \left| \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_0}}{\Gamma(1-p(t))} \right| \cdot |x_0(s)| ds + \int_0^{T_1} \left| \frac{(t-s)^{-p_0}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_0}}{\Gamma(1-p_0)} \right| \cdot |x_0(s)| ds \\
&\quad + \int_{T_1}^t \left| \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_1}}{\Gamma(1-p(t))} \right| \cdot |x_1(s)| ds + \int_{T_1}^t \left| \frac{(t-s)^{-p_1}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} \right| \cdot |x_1(s)| ds \\
&= \int_0^{T_1} \left| \frac{(t-s)^{-p(t^*)}}{\Gamma(1-p(t))} \left[(t-s)^{p(t^*)-p(t)} - (t-s)^{p(t^*)-p_0} \right] \right| \cdot |x_0(s)| ds \\
&\quad + \int_0^{T_1} \left| \frac{(t-s)^{-p_0}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_0}}{\Gamma(1-p_0)} \right| \cdot |x_0(s)| ds + \int_{T_1}^t \left| \frac{(t-s)^{-p_1}}{\Gamma(1-p(t))} - \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} \right| \cdot |x_1(s)| ds \\
&\quad + \int_{T_1}^t \left| \frac{(t-s)^{-p(t^*)}}{\Gamma(1-p(t))} \left[(t-s)^{p(t^*)-p(t)} - (t-s)^{p(t^*)-p_1} \right] \right| \cdot |x_1(s)| ds.
\end{aligned}$$

According to the inequalities (2.14), (2.15), we get that (2.13) holds for $t \in [0, T_2]$.

By the same arguments, we have

$$|I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-q(t)} x(t)| < \varepsilon, \quad 0 \leq t \leq T_i, \quad i = 3, 4, \dots, n^*, \quad T_{n^*} = T,$$

and

$$|I_{0+}^{1-p(t)} x(t) - I_{0+}^{1-q(t)} x(t)| < \varepsilon, \quad 0 \leq t < +\infty.$$

Hence, from the previous arguments, we could claim that (2.13) holds. Thus we complete this proof. \square

The following example illustrates that the semigroup property of the variable order fractional integral doesn't hold for the piecewise constant functions $p(t)$ and $q(t)$ defined in the same partition of finite interval $[a, b]$.

Example 2.8. Let $p(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ 3, & 1 < t \leq 4, \end{cases}$ $q(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 2, & 1 < t \leq 4, \end{cases}$ and $f(t) = 1, 0 \leq t \leq 4$. We'll verify $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}$.

$$\begin{aligned} & I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \\ = & \int_0^1 \frac{(t-s)^{2-1}}{\Gamma(2)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds + \int_1^t \frac{(t-s)^{3-1}}{\Gamma(3)} \int_0^s \frac{(s-\tau)^{2-1}}{\Gamma(2)} d\tau ds \\ = & \int_0^1 \frac{(t-s)s}{\Gamma(2)} ds + \int_1^t \frac{(t-s)^2 s^2}{2\Gamma(2)\Gamma(3)} ds, \end{aligned}$$

thus, we have

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} = \int_0^1 \frac{(3-s)s}{\Gamma(2)} ds + \int_1^3 \frac{(3-s)^2 s^2}{2\Gamma(2)\Gamma(3)} ds = \frac{83}{30}.$$

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=3} = \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(3)+q(3))} ds = \frac{3^{3+2}}{\Gamma(1+3+2)} = \frac{81}{40}.$$

Therefore, we obtain

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3},$$

which implies that semigroup property of the variable order fractional integral doesn't hold for the piecewise constant functions $p(t)$ and $q(t)$ defined in the same partition $[0, 1], (1, 4]$ of finite interval $[0, 4]$.

3. EXISTENCE OF APPROXIMATE SOLUTION

According to the previous arguments, we could obtain the explicit expression of solutions for the problem (1.1), nor, transform the problem (1.1) into an integral equation. Here, we consider the existence of the approximate solutions of the problem (1.1). In this section, we present our main results.

Now we make the following assumptions:

(A₂) For $0 \leq r \leq \{p_i, \rho\}$, $i = 0, 1, 2, \dots, n^* - 1$, $t^r f : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and there exist positive constants $c_1, c_2, 0 < \mu < 1, \lambda > \rho - r$, such that

$$t^r |f(t, (1 + t^\lambda)x(t))| \leq c_1 |x|^\mu + c_2, 0 \leq t < +\infty, x(t) \in \mathbb{R}.$$

let

$$E = \left\{ x(t) \mid x(t) \in C[0, +\infty), \sup_{t \geq 0} \frac{x(t)}{1 + t^\lambda} < \infty \right\}$$

with the norm

$$\|x\|_E = \sup_{t \geq 0} \frac{|x(t)|}{1 + t^\lambda},$$

where $\lambda > \rho - r$. Then, by the same arguments as in Lemma 2.2 [22], we know that $(E, \|\cdot\|_E)$ is a Banach space, here we omit this proof.

Lemma 3.1. Let M be a subset of E . Then M is precompact if and only if the following conditions are satisfied:

- (1) $\{\frac{x(t)}{1+t^\lambda} : x \in M\}$ is uniformly bounded;
- (2) $\{\frac{x(t)}{1+t^\lambda} : x \in M\}$ is equicontinuous in $[0, +\infty)$;
- (3) $\{\frac{x(t)}{1+t^\lambda} : x \in M\}$ equiconverges to zero as $t \rightarrow +\infty$, i.e., for all $\varepsilon > 0$, there exists a $L > 0$, such that for all $x \in M$ and $t \geq L$, $|\frac{x(t)}{1+t^\lambda}| < \varepsilon$.

Proof. We complete this proof by the same arguments as in Lemma 3.2.[22], here, we omit it. \square

Now, we consider the following initial value problem

$$\begin{cases} D_{0+}^{q(t)} x(t) = f(t, x), 0 < t < +\infty, \\ x(0) = 0, \end{cases} \quad (3.1)$$

where $q(t)$ is defined in (2.4).

In order to obtain our main results, we start off by carrying on essential analysis to the equation of (3.1).

By (2.4), we get

$$\int_0^t \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t))} x(s) ds = \sum_{k=0}^{n^*-1} I_k(t) \int_0^t \frac{(t-s)^{-p_k}}{\Gamma(1-p_k)} x(s) ds + I_\rho(t) \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x(s) ds.$$

So, the equation of (3.1) can be written by

$$\frac{d}{dt} \left(\sum_{k=0}^{n^*-1} I_k(t) \int_0^t \frac{(t-s)^{-p_k}}{\Gamma(1-p_k)} x(s) ds + I_\rho(t) \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x(s) ds \right) = f(t, x), \quad 0 < t < +\infty. \quad (3.2)$$

Then, equation (3.2) in the interval $(0, T_1]$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_0}}{\Gamma(1-p_0)} x(s) ds = D_{0+}^{p_0} x(t) = f(t, x), \quad 0 < t \leq T_1. \quad (3.3)$$

The equation (3.2) in the interval $(T_1, T_2]$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x(s) ds = f(t, x), \quad T_1 < t \leq T_2. \quad (3.4)$$

The equation (3.2) in the interval $(T_2, T_3]$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x(s) ds = f(t, x), \quad T_2 < t \leq T_3. \quad (3.5)$$

The equation (3.2) in the interval $(T_{i-1}, T_i], i = 4, 5, \dots, n^*$ ($T_{n^*} = T$) can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_{i-1}}}{\Gamma(1-p_{i-1})} x(s) ds = f(t, x), \quad T_{i-1} < t \leq T_i. \quad (3.6)$$

The equation (3.2) in the interval $(T, +\infty)$ can be written by

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x(s) ds = f(t, x), \quad T < t < +\infty. \quad (3.7)$$

Now, we present the definition of solution to the problem (3.1), which is critical in our work.

Definition 3.2. We say the problem (3.1) exists one solution, if there exist functions $u_i(t)$, $i = 0, 1, 2, \dots, n^* - 1$, such that $u_0 \in C[0, T_1]$ satisfying equation (3.3) and $u_0(0) = 0$; $u_1 \in C[0, T_2]$ satisfying equation (3.4) and $u_1(0) = 0$; $u_2 \in C[0, T_3]$ satisfying equation (3.5) and $u_2(0) = 0$; $u_{i-1} \in C[0, T_i]$ satisfying equation (3.6) and $u_{i-1}(0) = 0$ ($i = 4, 5, \dots, n^*$)($T_{n^*} = T$); $u_T \in C[0, +\infty)$ satisfying equation (3.7) and $u_T(0) = 0$.

The following is the definition of the approximate solution of the problem (1.1).

Definition 3.3. If there exist $T > 0$ and intervals $[0, T_1], (T_1, T_2], \dots, (T_{n^*-1}, T], (T, +\infty)$ ($n^* \in N$) and a function defined in (2.4), such that the problem (3.1) exists one solution, then, we say this solution of the problem (3.1) is one approximate solution of the problem (1.1).

Our main result is as following.

Theorem 3.4. Let conditions $(A_1), (A_2)$ hold, then the problem (1.1) has at least one approximate solution.

Proof. From Definitions 3.1, 3.2 and Lemma 2.4, we only need to consider the existence of solutions of the problem (3.1). According to the above analysis, equation of problem (3.1) can be written as the equation (3.2). The equation (3.2) in the interval $(0, T_1]$ can be written as (3.3). Applying operator $I_{0+}^{p_0}$ to both sides of (3.3), by Lemma 1.5, we have

$$x(t) = ct^{p_0-1} + \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} f(s, x(s)) ds, \quad 0 \leq t \leq T_1.$$

By $x(0) = 0$ and the assumption of function f , we get $c = 0$. Define operator $F : C[0, T_1] \rightarrow C[0, T_1]$ by

$$Fx(t) = \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} f(s, x(s)) ds, \quad 0 \leq t \leq T_1. \tag{3.8}$$

From the continuity of function $t^r f(t, x(t))$, we know that the operator $F : C[0, T_1] \rightarrow C[0, T_1]$ is well defined. In fact, let $g(t, x(t)) = t^r f(t, x(t))$, by (A_2) , one has $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. For $x(t) \in C[0, T_1]$, $t_0 \in [0, T_1]$, we have

$$\begin{aligned} & |Fx(t) - Fx(t_0)| \\ = & \left| \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} f(s, x(s)) ds - \frac{1}{\Gamma(p_0)} \int_0^{t_0} (t_0-s)^{p_0-1} f(s, x(s)) ds \right| \\ = & \left| \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} s^{-r} g(s, x(s)) ds - \frac{1}{\Gamma(p_0)} \int_0^{t_0} (t_0-s)^{p_0-s} s^{-r} g(s, x(s)) ds \right| \\ = & \left| \frac{t^{p_0-r}}{\Gamma(p_0)} \int_0^1 (1-\tau)^{p_0-1} \tau^{-r} g(t\tau, x(t\tau)) d\tau - \frac{t_0^{p_0-r}}{\Gamma(p_0)} \int_0^1 (1-\tau)^{p_0-1} \tau^{-r} g(t_0\tau, x(t_0\tau)) d\tau \right| \\ \leq & \frac{|t^{p_0-r} - t_0^{p_0-r}|}{\Gamma(p_0)} \int_0^1 (1-\tau)^{p_0-1} \tau^{-r} |g(t\tau, x(t\tau))| d\tau \\ & + \frac{t_0^{p_0-r}}{\Gamma(p_0)} \int_0^1 (1-\tau)^{p_0-1} \tau^{-r} |g(t\tau, x(t\tau)) - g(t_0\tau, x(t_0\tau))| d\tau. \end{aligned}$$

Together with continuity of functions g and t^{p_0-r} , we could easily obtain $Fx(t) \in C[0, T_1]$. By the standard arguments, $F : C[0, T_1] \rightarrow C[0, T_1]$ is a completely continuous operator.

Let $\Omega_1 = \{x \in C[0, T_1] \mid \|x\| \leq R_1\}$ be a closed and convex subset of $C[0, T_1]$, where

$$R_1 > \left\{ \frac{2c_2\Gamma(1-r)T_1^{p_0-r}}{\Gamma(1-r+p_0)}, \left(\frac{2c_1\Gamma(1-r)T_1^{p_0-r}}{\Gamma(1-r+p_0)} \right)^{\frac{1}{1-\mu}} \right\}.$$

For $x \in \Omega_1$, by (A₂), we get

$$\begin{aligned} |Fx(t)| &\leq \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} |f(s, x(s))| ds \\ &= \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} s^{-r} s^r |f(s, (1+s^\lambda)(1+s^\lambda)^{-1}x(s))| ds \\ &\leq \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} s^{-r} (c_1 \left(\frac{|x(s)|}{1+s^\lambda}\right)^\mu + c_2) ds \\ &\leq \frac{1}{\Gamma(p_0)} \int_0^t (t-s)^{p_0-1} s^{-r} (c_1 |x(s)|^\mu + c_2) ds \\ &\leq \frac{c_1\Gamma(1-r)T_1^{p_0-r}}{\Gamma(1-r+p_0)} R_1 R_1^{\mu-1} + \frac{c_2\Gamma(1-r)T_1^{p_0-r}}{\Gamma(1-r+p_0)} \\ &\leq \frac{R_1}{2} + \frac{R_1}{2} = R_1 \end{aligned}$$

which implies that $F(\Omega_1) \subseteq \Omega_1$. So the Schauder fixed point theorem assures that the operator F has one fixed point $x_0(t) \in C[0, T_1]$. Obviously, we could get $x_0(0) = 0$. So, $x_0(t)$ is one solution of the equation (3.3) with initial value condition $x(0) = 0$.

Also, we have obtained that the equation (3.2) in the interval $(T_1, T_2]$ can be written by (3.4). In order to consider the existence result of solutions to (3.4), we may discuss the following equation defined on interval $(0, T_2]$

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x(s) ds = D_{0+}^{p_1} x(t) = f(t, x), \quad 0 < t \leq T_2. \quad (3.9)$$

It is clear that if function $x \in C[0, T_2]$ satisfies the equation (3.9), then $x(t)$ must satisfy the equation (3.4). In fact, if $x^* \in C[0, T_2]$ with $x^*(0) = 0$ is a solution of the equation (3.9) with initial value condition $x(0) = 0$, that is

$$D_{0+}^{p_1} x^*(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x^*(s) ds = f(t, x^*), \quad 0 < t \leq T_2; \quad x^*(0) = 0.$$

Hence, from the equality above, we have that $x^* \in C[0, T_2]$ with $x^*(0) = 0$ satisfies the equation

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-p_1}}{\Gamma(1-p_1)} x^*(s) ds = f(t, x^*), \quad T_1 \leq t \leq T_2,$$

which means the function $x^* \in [0, T_2]$ with $x^*(0) = 0$ is a solution of the equation (3.4).

Based on this fact, we consider the existence of solutions to the equation (3.9) with initial value condition $x(0) = 0$.

Now, applying operator $I_{0+}^{p_1}$ on both sides of (3.9), by Lemma 1.5, we have

$$x(t) = ct^{p_1-1} + \frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} f(s, x(s)) ds, \quad 0 \leq t \leq T_2.$$

By initial value condition $x(0) = 0$, we have $c = 0$.

Define operator $F : C[0, T_2] \rightarrow C[0, T_2]$ by

$$Fx(t) = \frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} f(s, x(s)) ds, \quad 0 \leq t \leq T_2.$$

From the previous arguments, the operator $F : C[0, T_2] \rightarrow C[0, T_2]$ is a completely continuous operator.

Let $\Omega_2 = \{x \in C[0, T_2] \mid \|x\| \leq R_2\}$ be a closed and convex subset of $C[0, T_2]$, where

$$R_2 > \left\{ \frac{2c_2\Gamma(1-r)T_2^{p_1-r}}{\Gamma(1-r+p_1)}, \left(\frac{2c_1\Gamma(1-r)T_2^{p_1-r}}{\Gamma(1-r+p_1)} \right)^{\frac{1}{1-\mu}} \right\}.$$

For $x \in \Omega_2$, by (A₂), we get that

$$\begin{aligned} |Fx(t)| &\leq \frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} s^{-r} \left(c_1 \left(\frac{|x(s)|}{1+s^\lambda} \right)^\mu + c_2 \right) ds \\ &\leq \frac{c_1\Gamma(1-r)T_2^{p_1-r}}{\Gamma(1-r+p_1)} R_2^\mu + \frac{c_2\Gamma(1-r)T_2^{p_1-r}}{\Gamma(1-r+p_1)} \\ &\leq \frac{R_2}{2} + \frac{R_2}{2} = R_2 \end{aligned}$$

which implies that $F(\Omega_2) \subseteq \Omega_2$. So the Schauder fixed point theorem assures that the operator F has one fixed point $x_1(t) \in \Omega_2 \subset C[0, T_2]$, that is,

$$x_1(t) = \frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} f(s, x_1(s)) ds, \quad 0 \leq t \leq T_2. \tag{3.10}$$

From (3.10), we know $x_1(0) = 0$. Applying operator $D_{0+}^{p_1}$ on both sides of (3.10), by Lemma 1.2, we obtain that

$$D_{0+}^{p_1} x_1(t) = f(t, x_1), \quad 0 < t \leq T_2,$$

that is, $x_1(t)$ satisfies the equation as following

$$\frac{d}{dt} \frac{1}{\Gamma(1-p_1)} \int_0^t (t-s)^{-p_1} x_1(s) ds = f(t, x_1(t)), \quad 0 < t \leq T_2; \quad x_1(0) = 0.$$

From the previous arguments, we obtain $x_1 \in C[0, T_2]$ with $x_1(0) = 0$ satisfies the equation (3.4).

By the similar way, for $i = 3, \dots, n^*$, we get that the equation (3.2) defined on $(T_{i-1}, T_i]$ has one solution $x_{i-1}(t) \in \Omega_i \subset C[0, T_i]$ with $x_{i-1}(0) = 0$ ($T_{n^*} = T$), where $\Omega_i = \{u \in C[0, T_i] \mid \|u\| \leq R_i\}$ is closed and the convex subset of $C[0, T_i]$, R_i satisfies

$$R_i > \left\{ \frac{2c_2\Gamma(1-r)T_i^{p_{i-1}-r}}{\Gamma(1-r+p_{i-1})}, \left(\frac{2c_1\Gamma(1-r)T_i^{p_{i-1}-r}}{\Gamma(1-r+p_{i-1})} \right)^{\frac{1}{1-\mu}} \right\}.$$

Finally, we obtain the equation (3.2) in the interval $(T, +\infty)$ can be written by (3.7). In order to consider the existence result of solutions to (3.7), we may discuss the following equation defined on interval $(0, +\infty)$

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x(s) ds = D_{0+}^\rho x(t) = f(t, x), \quad 0 < t < +\infty. \quad (3.11)$$

With function $x \in C[0, +\infty)$ satisfies the equation (3.11), $x(t)$ must satisfy the equation (3.7). In fact, if $x^* \in C[0, +\infty)$ with $x^*(0) = 0$ is a solution of the equation (3.11) with initial value condition $x(0) = 0$, that is

$$D_{0+}^\rho x^*(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x^*(s) ds = f(t, x^*), \quad 0 < t < +\infty; \quad x^*(0) = 0.$$

Hence, from the equality above, we have $x^* \in C[0, +\infty)$ with $x^*(0) = 0$ satisfies the equation

$$\frac{d}{dt} \int_0^t \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)} x(s) ds = f(t, x), \quad T < t < +\infty,$$

which means the function $x^* \in C[0, +\infty)$ with $x^*(0) = 0$ is a solution of the equation (3.7).

Based on this fact, we will consider the existence of solutions to the equation (3.11) with initial value condition $x(0) = 0$.

Now, applying operator I_{0+}^ρ on both sides of (3.11), by Lemma 1.5, we have that

$$x(t) = ct^{\rho-1} + \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x(s)) ds, \quad 0 \leq t < +\infty.$$

By initial value condition $x(0) = 0$, we have $c = 0$. Define the operator $F : E \rightarrow E$ by

$$Fx(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x(s)) ds, \quad 0 \leq t < +\infty.$$

To get the operator $F : E \rightarrow E$ is well defined. First, we verify that $Fx(t) \in C[0, +\infty)$ for $x \in E$. In fact, let

$$g(t, x) = t^r f(t, x),$$

by (A_2) , we know that $g : [0, +\infty) \times R \rightarrow R$ is continuous.

For the case of $t_0 \in (0, +\infty)$, take $t > t_0$, $t - t_0 < 1$, then

$$(t_0 - s)^{\rho-1} > (t - s)^{\rho-1}, \quad 0 \leq s < t_0.$$

$$\begin{aligned} & \left| \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x(s)) ds - \frac{1}{\Gamma(\rho)} \int_0^{t_0} (t_0-s)^{\rho-1} f(s, x(s)) ds \right| \\ & \leq \frac{c_1 \|x\|_E^\mu + c_2}{\Gamma(\rho)} \left[\int_0^{t_0} ((t_0-s)^{\rho-1} - (t-s)^{\rho-1}) s^{-r} ds \right. \\ & \quad \left. + \int_{t_0}^t (t-s)^{\rho-1} s^{-r} ds \right]. \end{aligned}$$

We will consider the two terms in brackets respectively. It is easy to show that

$$\int_0^t (t-s)^{\rho-1} s^{-r} ds = t^{\rho-r} \beta(\rho, 1-r).$$

Hence, for any given $\varepsilon > 0$, there exists a $\delta_1 > 0$, such that, if $0 \leq t \leq \delta_1$, it holds that

$$\left| \int_0^t (t-s)^{\rho-1} s^{-r} ds \right| < \frac{\varepsilon}{4}. \quad (3.12)$$

Moreover, we get

$$\begin{aligned} & \int_{\delta_1}^{t_0} ((t_0-s)^{\rho-1} - (t-s)^{\rho-1}) s^{-r} ds \\ & \leq \delta_1^{-r} \int_{\delta_1}^{t_0} ((t_0-s)^{\rho-1} - (t-s)^{\rho-1}) ds \\ & = \frac{\delta_1^{-r}}{\rho} ((t_0-\delta_1)^\rho - (t-\delta_1)^\rho + (t-t_0)^\rho) \\ & \leq \frac{\delta_1^{-r}}{\rho} (t-t_0)^\rho, \end{aligned}$$

hence, we know that there exists $\delta_2 > 0$ such that for $0 < t - t_0 < \delta_2$, we have

$$\int_{\delta_1}^{t_0} ((t_0-s)^{\rho-1} - (t-s)^{\rho-1}) s^{-r} ds < \frac{\varepsilon}{2},$$

together with (3.12), it leads to

$$\int_0^{t_0} ((t_0-s)^{\rho-1} - (t-s)^{\rho-1}) s^{-r} ds < \varepsilon.$$

For the second brackets, by the direct calculation, we have

$$\int_{t_0}^t (t-s)^{\rho-1} s^{-r} ds \leq t_0^{-r} \frac{(t-t_0)^\rho}{\rho},$$

which implies that there exists $\delta_3 > 0$ such that for $0 < t - t_0 < \delta_3$, we get

$$\int_{t_0}^t (t-s)^{\rho-1} s^{-r} ds < \varepsilon.$$

Combining with these analysis above, we obtain $\int_0^t (t-s)^{\rho-1} f(s, x(s)) ds$ is continuous on point t_0 . In view of the arbitrariness of t_0 , we have $Fx \in C(0, +\infty)$.

For the case of $t_0 = 0$, by (A_2) , we obtain the continuity of $Fx(t)$.

For $x \in E$, by (A_2) , we have

$$\begin{aligned} \left| \frac{Fx(t)}{1+t^\lambda} \right| & \leq \frac{1}{(1+t^\lambda)\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} s^{-r} (c_1 \left(\frac{|x(s)|}{1+s^\lambda} \right)^\mu + c_2) ds \\ & \leq \frac{c_1}{(1+t^\lambda)\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} s^{-r} \|x\|_E^\mu ds + \frac{C_2\Gamma(1-r)}{(1+t^\lambda)\Gamma(1-r+\rho)} t^{\rho-r} \\ & \leq \frac{c_1 \|x\|_E^\mu \Gamma(1-r) t^{\rho-r}}{(1+t^\lambda)\Gamma(1-r+\rho)} + \frac{C_2\Gamma(1-r) t^{\rho-r}}{(1+t^\lambda)\Gamma(1-r+\rho)}, \end{aligned}$$

which implies that $\lim_{t \rightarrow +\infty} \frac{Fx(t)}{1+t^\lambda} = 0$.

Hence, $F : E \rightarrow E$ is well defined.

Second, we prove that the operator F is continuous. Let $\{x_n\} \subset E$ and $\lim_{n \rightarrow \infty} \|x_n - x\|_E = 0$, we will prove: for $\forall \varepsilon > 0$, there exists $N > 0$ such that for $n \geq N$, $\|Fx_n - Fx\|_E < \varepsilon$.

Since $\lim_{n \rightarrow \infty} \|x_n - x\|_E = 0$, there exists $K > 0$ such that $\|x_n\|_E \leq K$ ($n = 1, 2, \dots$) and $\|x\|_E \leq K$.

By (A_2) , for any n , we have

$$\begin{aligned} & \left| \frac{1}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} (f(s, x_n(s)) - f(s, x(s))) ds \right| \\ & \leq \frac{2c_1 K^\mu}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} s^{-r} ds + \frac{2c_2}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} s^{-r} ds \\ & = \frac{2c_1 K^\mu \Gamma(1-r)}{(1+t^\lambda) \Gamma(1-r+\rho)} t^{\rho-r} + \frac{2C_2 \Gamma(1-r)}{(1+t^\lambda) \Gamma(1-r+\rho)} t^{\rho-r}. \end{aligned}$$

Hence, for $\forall \varepsilon > 0$, there exist constant $N_1 > 0$ and $\delta > 0$, such that

$$\left| \frac{1}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} (f(s, x_n(s)) - f(s, x(s))) ds \right| < \varepsilon, t \geq N_1, \quad (3.13)$$

$$\left| \frac{1}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} (f(s, x_n(s)) - f(s, x(s))) ds \right| < \varepsilon, 0 \leq t \leq \delta. \quad (3.14)$$

As well, from

$$\lim_{n \rightarrow \infty} \sup_{\delta \leq s \leq N_1} \frac{|x_n(s) - x(s)|}{1+s^\lambda} \leq \lim_{n \rightarrow \infty} \sup_{s \geq 0} \frac{|x_n(s) - x(s)|}{1+s^\lambda} = \lim_{n \rightarrow \infty} \|x_n - x\|_E = 0,$$

we get $\lim_{n \rightarrow \infty} \sup_{\delta \leq s \leq N_1} |x_n(s) - x(s)| = 0$. Thus, from the uniform continuity of f on any compact subset, we could get

$$\lim_{n \rightarrow \infty} \sup_{\delta \leq s \leq N_1} |f(s, x_n(s)) - f(s, x(s))| = 0$$

Hence, when $\delta \leq t \leq N_1$, there exists $N > 0$ such that for any $n \geq N$,

$$\begin{aligned} & \left| \frac{1}{\Gamma(\rho)} \int_\delta^{N_1} \frac{(t-s)^{\rho-1}}{1+t^\lambda} (f(s, x_n(s)) - f(s, x(s))) ds \right| \\ & \leq \sup_{\delta \leq s \leq N_1} |f(s, x_n(s)) - f(s, x(s))| \sup_{\delta \leq t \leq N_1} \frac{(t-\delta)^\rho - (t-N_1)^\rho}{\Gamma(1+\rho)(1+t^\lambda)} \\ & \leq \sup_{\delta \leq s \leq N_1} |f(s, x_n(s)) - f(s, x(s))| \sup_{\delta \leq t \leq N_1} \frac{1}{\Gamma(1+\rho)} < \varepsilon. \end{aligned}$$

Together with (3.13), (3.14), then, F is continuous. Finally, we will verify F is a compact operator by Lemma 3.1. Let $M \subset E$ be a bounded set. So there exists $W > 0$ such that $\|x\|_E \leq W$ for $x \in M$. Obviously, we only need to prove FM satisfy the conditions in Lemma 3.1. Considering the previous arguments, we easily prove that (1) and (3) satisfied. For any $x \in M$ and $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$, we have

$$\begin{aligned} \left| \frac{Fx(t_1)}{1+t_1^\lambda} - \frac{Fx(t_2)}{1+t_2^\lambda} \right| & \leq \frac{1}{\Gamma(\rho)} \left| \int_0^{t_1} \left(\frac{(t_1-s)^{\rho-1}}{1+t_1^\lambda} - \frac{(t_2-s)^{\rho-1}}{1+t_2^\lambda} \right) f(s, x(s)) ds \right| \\ & \quad + \frac{1}{\Gamma(\rho)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\rho-1}}{1+t_2^\lambda} |f(s, x(s))| ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{c_1 W^\mu}{\Gamma(\rho)} \left| \int_0^{t_1} \left(\frac{(t_1-s)^{\rho-1}}{1+t_1^\lambda} - \frac{(t_2-s)^{\rho-1}}{1+t_2^\lambda} \right) s^{-r} ds \right| \\ &\quad + \frac{c_2}{\Gamma(\rho)} \left| \int_0^{t_1} \left(\frac{(t_1-s)^{\rho-1}}{1+t_1^\lambda} - \frac{(t_2-s)^{\rho-1}}{1+t_2^\lambda} \right) s^{-r} ds \right| \\ &\quad + \frac{c_1 W^\mu}{\Gamma(\rho)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\rho-1}}{1+t_2^\lambda} s^{-r} ds + \\ &\quad + \frac{c_2}{\Gamma(\rho)} \int_{t_1}^{t_2} \frac{(t_2-s)^{\rho-1}}{1+t_2^\lambda} s^{-r} ds, \end{aligned}$$

by similar arguments as before, we get $\{Fx(t)/(1+t^\lambda) : x \in M\}$ is equicontinuous. Hence, F is compact. As a result, $F : E \rightarrow E$ is a completely continuous operator. Let $\Omega = \{x \in E : \|x\|_E \leq R\}$ be closed and convex subset of E , where

$$R > \left\{ \left(\frac{2c_1\Gamma(1-r)}{\Gamma(1-r+\rho)} \right)^{\frac{1}{1-\mu}}, \frac{2c_2\Gamma(1-r)}{\Gamma(1-r+\rho)} \right\}.$$

For $x \in \Omega$, from (A_2) , we have

$$\begin{aligned} \left| \frac{Fx(t)}{1+t^\lambda} \right| &\leq \frac{1}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} |f(s, x(s))| ds \\ &\leq \frac{c_1}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} s^{-r} \|x\|_E^\mu ds + \frac{c_2}{\Gamma(\rho)} \int_0^t \frac{(t-s)^{\rho-1}}{1+t^\lambda} s^{-r} ds \\ &\leq \frac{c_1\Gamma(1-r)t^{\rho-r}}{\Gamma(1-r+\rho)} \frac{t^{\rho-r}}{1+t^\lambda} RR^{\mu-1} + \frac{c_2\Gamma(1-r)}{\Gamma(1-r+\rho)} \frac{t^{\rho-r}}{1+t^\lambda} \\ &\leq \frac{c_1\Gamma(1-r)}{\Gamma(1-r+\rho)} RR^{\mu-1} + \frac{c_2\Gamma(1-r)}{\Gamma(1-r+\rho)} \\ &\leq \frac{R}{2} + \frac{R}{2} = R, \end{aligned}$$

which implies that $F(\Omega) \subseteq \Omega$. So the Schauder fixed point theorem assures that the operator F has one fixed point $x_T(t) \in \Omega \subset E$. And, $x_T \in E$ is a solution of the equation (3.7) can be proved by the similar methods. Thus, according to Definition 3.2, we obtain that the problem (1.1) has one approximate solution. \square

Example 3.5. Now, we consider the initial value problem as following

$$D_{0+}^{\frac{t}{e^{t+\frac{1}{100}}}} x(t) = \frac{x^{\frac{2}{3}}}{(1+t^2)^{\frac{2}{3}}(1+x^4)}, 0 < t < +\infty, x(0) = 0, \tag{3.15}$$

We let $p(t) = \frac{t}{e^{t+\frac{1}{100}}}, 0 \leq t < +\infty$ and $f(t, x) = \frac{x^{\frac{2}{3}}}{(1+t^2)^{\frac{2}{3}}(1+x^4)}, 0 < t < +\infty, x \in R$. Obviously,

$$|f(t, (1+t^2)x)| \leq \frac{x^{\frac{2}{3}}}{1+(1+t^2)^4x^4} \leq x^{\frac{2}{3}}, 0 \leq t < +\infty, x \in R,$$

which implies that f satisfies (A_2) with $r = 0, \mu = \frac{2}{3}, c_1 = 1, c_2 = 1$. And that, it holds $\lim_{t \rightarrow +\infty} p(t) = 0$, thus, p satisfies (A_1) . For given arbitrary small $\varepsilon = \frac{1}{100}$,

there exists $T = \frac{2.2}{\varepsilon} = 200$, such that

$$|p(t)| = \frac{t}{e^{t+\frac{1}{100}}} < \frac{t}{e^t} < \frac{t}{\frac{t^3}{3}} = \frac{3}{t^2} \leq \frac{3\varepsilon^2}{4} < \frac{3\varepsilon}{4}, t \geq T.$$

Let $\rho = \frac{\varepsilon}{4} = \frac{1.1}{400}$, it holds

$$|p(t) - \rho| \leq |p(t)| + \rho < \frac{3\varepsilon}{4} + \rho = \varepsilon, t \geq T.$$

Now, we consider function $p(t)$ restricted on interval $[0, T] = [0, 200]$. By the right continuity of function $p(t)$ at point 0, for $\varepsilon = \frac{1.1}{100}$, taking $\delta_0 = \frac{1}{100}$, when $0 \leq t \leq \delta_0 = \frac{1}{100}$, we have

$$|p(t) - p(0)| = \left| \frac{t}{e^{t+\frac{1}{100}}} \right| \leq t \leq \delta_0 < \frac{1.1}{100} = \varepsilon.$$

We get $t_1 = \delta_0 = \frac{1}{100}$. By the right continuity of function $p(t)$ at point t_1 , for $\varepsilon = \frac{1.1}{100}$, taking $\delta_1 = \frac{1}{100}$, when $0 \leq t - t_1 \leq \delta_1$, by differential mean value theorem, we have

$$\begin{aligned} |p(t) - p(t_1)| &= \left| \frac{t}{e^{t+\frac{1}{100}}} - \frac{t_1}{e^{t_1+\frac{1}{100}}} \right| \\ &\leq |(1 - \xi)e^{-(\xi+\frac{1}{100})}| |t - t_1| \\ &\leq \frac{1 + t}{e^{t_1+\frac{1}{100}}} |t - t_1| \\ &< \frac{1 + t_1 + \delta_1}{e^{t_1+\frac{1}{100}}} |t - t_1| \\ &= \frac{1 + t_1 + \frac{1}{100}}{e^{t_1+\frac{1}{100}}} |t - t_1| \\ &< |t - t_1| \leq \delta_1 < \varepsilon = \frac{1.1}{100}, \end{aligned}$$

where $t_1 < \xi < t$. Continuing this procession, from $t_{n-1} = \frac{n-1}{100} < 200$, $t_n = t_{n-1} + \delta_{n-1} = \frac{n-1}{100} + \frac{1}{100} = \frac{n}{100}$, we get $n = 20000$. Thus, let

$$p_0 \doteq p(0) = 0, p_1 \doteq p(t_1) = p\left(\frac{1}{100}\right) = \frac{1}{100e^{\frac{2}{100}}}, p_2 \doteq p(t_2) = p\left(\frac{2}{100}\right) = \frac{2}{100e^{\frac{3}{100}}}, \dots,$$

$$p_{19998} \doteq p(t_{19998}) = p\left(\frac{19998}{100}\right) = \frac{19998}{100e^{\frac{19999}{100}}}, p_{19999} \doteq p(t_{19999}) = p\left(\frac{19999}{100}\right) = \frac{19999}{100e^{\frac{20000}{100}}}.$$

As a result, we get intervals $[0, \frac{1}{100}), [\frac{1}{100}, \frac{2}{100}), \dots, [199.98, 199.99), [199.99, 200], (200, +\infty)$ and function $q(t)$ defined by

$$q(t) = \begin{cases} p_0 = 0, & \text{for } t \in [0, \frac{1}{100}], \\ p_1 = \frac{1}{100e^{\frac{2}{100}}}, & \text{for } t \in (\frac{1}{100}, \frac{2}{100}], \\ p_2 = \frac{2}{100e^{\frac{3}{100}}}, & \text{for } t \in (\frac{2}{100}, \frac{3}{100}], \\ \dots, \\ p_{19998} = \frac{19998}{100e^{\frac{19999}{100}}}, & \text{for } t \in (199.98, 199.99], \\ p_{19999} = \frac{19999}{100e^{\frac{20000}{100}}}, & \text{for } t \in (199.99, 200] \\ \rho = \frac{1.1}{400}, & \text{for } t \in (200, +\infty). \end{cases}$$

By Definitions 3.1, 3.2 and the arguments of Theorem 3.1, the problem (3.15) has one approximate solution.

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