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MULTIPLE MELLIN AND LAPLACE TRANSFORMS OF I-FUNCTIONS OF r VARIABLES

B.K. DUTTA, L.K. ARORA

Abstract. The aim of this paper is to study multiple Mellin and Laplace transforms involving multivariable I-function. In this regard, we have proved five theorems, and a few corollaries have also been recorded. Similar results for the H-function of two and r variables obtained by other authors follow as special case of our findings.

1. Introduction

In recent years, many authors pointed out that derivatives and integrals of fractional order are suitable for description of properties of various real materials. It has proved to be very useful tool for modeling of many phenomena in physics, chemistry, engineering, bioscience and other areas. The main advantage of fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Whereas the classical integer order models neglected such effects. For more details, we refer to [3, 4, 5, 8, 9, 12, 13, 16]. The Mellin transform may be regarded as the multiplicative version of the two-sided Laplace transform. It is closely connected to Laplace transform, Fourier transform, theory of the gamma function and allied special functions. The problem of derivation the exact solutions for fractional differential equations with mixed derivatives is an important and emerging area in fractional calculus. The Mellin Transform is widely used in fractional calculus because of its scale invariance property [3]. Klimek, Dziembowski [6] and Klimek [7] proposed to apply the Mellin transform method for fractional differential equations.
Recently, the multivariable I-function has been introduced and studied by Prasad [17] and Prasad and Yadav [18], which is a generalization of multivariable H-function. Further, Prasad and Singh [14] studied the Mellin and Laplace transform of the multivariable I-function. They observed that the Mellin transform of I-function of r variables reduces to the I-function of $r-1$ variables. The derivatives of multivariable I-function has been studied by Saxena and Singh [10]. Chaurasia and Kumar [15] investigated the fractional integrals of product of $\bar{H}$-function [1] and mutivariable I-function.

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In this paper, we have studied various multiple Mellin and Laplace transform of I-function. Since the multivariable I-function is of general nature, its multiple Mellin and Laplace transforms reduces to many simpler special functions as particular cases.

For convenience, we first recall some definitions and fundamental facts of integral transform and special functions.

Definition 1.1 The Mellin transform of a function $\varphi(t)$, $t > 0$, is defined (see [3] and [12]) as follows

$$\varphi^*(s) = \mathcal{M}\{\varphi; s\} = \int_0^\infty t^{s-1}\varphi(t)dt, \tag{1}$$

and if the function $\varphi(t)$ also satisfies the Dirichlet conditions every closed interval $[a,b] \subset (0,\infty)$, then the function $\varphi(t)$ can be restored using inverse Mellin transform formula

$$\varphi(x) = \mathcal{M}^{-1}\{\varphi^*(s); x\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \varphi^*(s)x^{-s}ds, \ (0 < x < \infty) \tag{2}$$

where s is a complex, such that $\gamma_1 < \Re(s) < \gamma_2$ and $\gamma_1 < \gamma < \gamma_2$.
The Mellin transform (1) exists if the function $\varphi(t)$ is piecewise continuous in every closed interval $[a,b] \subset (0,\infty)$ and

$$\int_0^1 |\varphi(t)|t^{\gamma_1-1}dt < \infty, \quad \int_1^\infty |\varphi(t)|t^{\gamma_2-1}dt < \infty. \tag{3}$$

Definition 1.2 The multidimensional Mellin transform for the function $\varphi(t) = \varphi(t_1, \cdots, t_r)$ define as follows [12]

$$\begin{aligned}(\mathfrak{M}\varphi)(s) :&= \int_{(R)^n_{+\cdots+}} t^{s-1}\varphi(t)dt \\ &= \int_0^\infty \cdots \int_0^\infty t_1^{s_1-1}\cdots t_r^{s_r-1}\varphi(t_1,\cdots,t_r)dt_1\cdots dt_r,\end{aligned} \tag{4}$$

where $t_1, \cdots, t_r > 0$.

Definition 1.3 The multidimensional Laplace transform define for the function $f(x_1, \cdots, x_r)$ as follows

$$\begin{aligned}(\mathfrak{L}f)(s) :&= \mathfrak{L}\{f(x_1,\cdots,x_r); s_1,\cdots,s_r\} \\ &= \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^r s_i x_i\right) f(x_1,\cdots,x_r)dx_1\cdots dx_r,\end{aligned} \tag{5}$$

where $\Re(s_i) > 0, i \in \{1, \cdots, r\}$.

Definition 1.4 The *multivariable I-function* represent [17] as

$$\begin{aligned}I[z_1,\cdots,z_r] :&= I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)})\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)},q^{(i)})\}^{1,r}}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\left|\begin{matrix}\mathcal{A}:\mathcal{B} \\ \mathcal{C}:\mathcal{D}\end{matrix}\right.\right] \\ &= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\zeta_1,\cdots,\zeta_r)\prod_{i=1}^r\left\{\phi_i(\zeta_i)z_i^{\zeta_i}\right\}d\zeta_1\cdots d\zeta_r\end{aligned} \tag{6}$$

where $\omega = \sqrt{-1}$,

$$\psi(\zeta_1,\cdots,\zeta_r)=\frac{\prod_{k=2}^{r}\left[\prod_{j=1}^{n_k}\Gamma(1-a_{kj}+\sum_{i=1}^{k}\alpha_{kj}^{(i)}\zeta_i)\right]}{\prod_{k=2}^{r}\left[\prod_{j=n_k+1}^{p_k}\Gamma(a_{kj}-\sum_{i=1}^{k}\alpha_{kj}^{(i)}\zeta_i)\right]} \times\frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_k}\Gamma(1-b_{kj}+\sum_{i=1}^{k}\beta_{kj}^{(i)}\zeta_i)\right]}, \tag{7}$$

$$\phi_i(\zeta_i)=\frac{\left[\prod_{k=1}^{m^{(i)}}\Gamma(b_k^{(i)}-\beta_k^{(i)}\zeta_i)\right]\left[\prod_{j=1}^{n^{(i)}}\Gamma(1-a_j^{(i)}+\alpha_j^{(i)}\zeta_i)\right]}{\left[\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a_j^{(i)}-\alpha_j^{(i)}\zeta_i)\right]\left[\prod_{k=m^{(i)}+1}^{q^{(i)}}\Gamma(1-b_k^{(i)}+\beta_k^{(i)}\zeta_i)\right]}, \tag{8}$$

$\forall i\in\{1,\cdots,r\}$. Also,

$$\begin{aligned}
&\{0,n_i\}_{2,r} &&:=0,n_2:\cdots:0,n_r,\\
&\{p_i,q_i\}_{2,r} &&:=p_2,q_2:\cdots:p_r,q_r,\\
&\left\{\left(m^{(i)},n^{(i)}\right)\right\}^{1,r} &&:=(m^{(1)},n^{(1)});\cdots;(m^{(r)},n^{(r)}),\\
&\left\{\left(p^{(i)},q^{(i)}\right)\right\}^{1,r} &&:=(p^{(1)},q^{(1)});\cdots;(p^{(r)},q^{(r)}),\\
\mathcal{A}:=:&\left\{\left(a_{ij};\alpha_{ij}^{(1)},\cdots,\alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}\right\} &&:=(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)})_{1,p_2};\cdots;(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)})_{1,p_r},\\
\mathcal{B}:=:&\left\{\left(a_j^{(i)},\alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}\right\} &&:=(a_j^{(1)},\alpha_j^{(1)})_{1,p^{(1)}};\cdots;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}},\\
\mathcal{C}:=:&\left\{(b_{ij};\beta_{ij}^{(1)},\cdots,\beta_{ij}^{(i)})_{1,q_i}^{2,r}\right\} &&:=(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)})_{1,q_2};\cdots;(b_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)})_{1,q_r},\\
\mathcal{D}:=:&\left\{\left(b_j^{(i)},\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}\right\} &&:=(b_j^{(1)},\beta_j^{(1)})_{1,q^{(1)}};\cdots;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}
\end{aligned}$$

such that $n_i,p_i,q_i,m^{(i)},n^{(i)},p^{(i)},q^{(i)}$ are non-negative integers and all $a_{ij},b_{ij},\alpha_{ij}$, $\beta_{ij},a_j^{(i)},b_j^{(i)},\alpha_j^{(i)},\beta_j^{(i)}$ are complex numbers and the empty product denotes unity. The contour integral (6) converges, if

$$|\arg z_i|<\frac{1}{2}U_i\pi,\ U_i>0,\ i=1,\cdots,r, \tag{9}$$

where

$$U_i=\sum_{j=1}^{n^{(i)}}\alpha_j^{(i)}-\sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)}+\sum_{j=1}^{m^{(i)}}\beta_j^{(i)}-\sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)}+\left(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)}-\sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}\right) +\cdots+\left(\sum_{j=1}^{n_r}\alpha_{rj}^{(i)}-\sum_{j=n_r+1}^{p_r}\alpha_{rj}^{(i)}\right)-\left(\sum_{j=1}^{q_2}\beta_{2j}^{(i)}+\cdots+\sum_{j=1}^{q_r}\beta_{rj}^{(i)}\right) \tag{10}$$

and $I[z_1,\cdots,z_r]=O(|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r})$, $\max\{|z_1|,\cdots,|z_r|\}\to 0$,
where $\alpha_i=\min_{1\le j\le m^{(i)}}\Re(b_j^{(i)}/\beta_j^{(i)})$, and $\beta_i=\max_{1\le j\le n^{(i)}}\Re((a_j^{(i)}-1)/\alpha_j^{(i)})$, $i=1,\cdots,r$.

For the condition of convergence and analyticity of multivariable I-function we refer [17, 18].

2. Main Results

The main results to be established here as follows

Theorem 2.1 Suppose the conditions (9) to be satisfied. The Multiple Mellin transform of I-function of r variable define in (6) as follows

$$(\mathfrak{M}I)(s) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r t_i^{s_i-1} I[z_1 t_1^{\mu_1}, \cdots, z_r t_r^{\mu_r}] dt_1 \cdots dt_r$$
$$= \prod_{i=1}^r \left\{ \frac{z_i^{-\frac{s_i}{\mu_i}}}{\mu_i} \phi_i \left(-\frac{s_i}{\mu_i} \right) \right\} \psi \left(-\frac{s_1}{\mu_1}, \cdots, -\frac{s_r}{\mu_r} \right) \tag{11}$$

where

$$-\mu_i \min_{1 \le j \le m^{(i)}} \Re \left(\frac{b_j^{(i)}}{\beta_j^{(i)}} \right) < \Re(s_i) < \mu_i \min_{1 \le j \le n^{(i)}} \Re \left(\frac{1 - a_j^{(i)}}{\alpha_j^{(i)}} \right),$$

$\psi(t_1, \cdots, t_r)$ and $\phi(t_i)$, $i \in \{1, \cdots, r\}$ are given in (7) and (8) respectively.
Proof First we express the multivariable I-function in left side of the integration of (11) as a product of multiple Mellin-Barnes contour integral by using (6) and interchanging the order of integration, which is permissible under the above stated conditions. After a straightforward calculation we finally arrived at (11). ■

Corollary 2.1. *When $n_i = 0, p_i = 0, q_i = 0, i = 2, \cdots, r-1$ (the empty product denotes unity) the* (11) *reduces to the multiple Mellin transformation of H-function of r variables* [2, p. 251].

Corollary 2.2. *When $r = 2$ and $n_2 = 0,\ p_2 = 0,\ q_2 = 0$ in* (11)*, it reduces to the double Mellin transformation of H-function of two variables* [2, p. 147].

Theorem 2.2 Suppose the conditions (9) to be satisfied. Then Multiple Laplace transform of multivariable I-function is as follows

$$(\mathfrak{L}I)(s) = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^r s_i t_i \right) \prod_{i=1}^r t_i^{\rho_i - 1} I[z_1 t_1^{\mu_1}, \cdots, z_r t_r^{\mu_r}] dt_1 \cdots dt_r$$
$$= \prod_{i=1}^r \left\{ s_i^{-\rho_i} \right\} I^{\{0,n_i\}_{2,r}:\left\{(m^{(i)},n^{(i)}+1)\right\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\left\{(p^{(i)}+1,q^{(i)})\right\}^{1,r}} \left[\begin{matrix} z_1 s_1^{-\mu_1} \\ \vdots \\ z_r s_r^{-\mu_r} \end{matrix} \middle| \begin{matrix} \mathcal{A} : \mathcal{B} \\ \mathcal{C} : \mathcal{D} \end{matrix} \right] \tag{12}$$

where $\mu_i > 0, \Re(s_i) > 0, \Re(\rho_i) + \mu_i \min_{1 \le j \le m^{(i)}} \Re \left(\frac{b_j^{(i)}}{\beta_j^{(i)}} \right) > 0,\ i = 1, \cdots, r,$

$$\mathcal{A} \equiv \left\{ (a_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)})_{1,p_i}^{2,r} \right\},\ \mathcal{B} \equiv \left\{ (1-\rho_i, \mu_i), \left(a_j^{(i)}, \alpha_j^{(i)} \right)_{1,p^{(i)}}^{1,r} \right\},$$
$$\mathcal{C} \equiv \left\{ (b_{rj}; \beta_{rj}^{(1)}, \cdots, \beta_{rj}^{(r)})_{1,q_i}^{2,r} \right\},\ \mathcal{D} \equiv \left\{ \left(b_j^{(i)}, \beta_j^{(i)} \right)_{1,q^{(i)}}^{1,r} \right\}$$

Proof First we express the multivariable I-function on left side of the integration of (12) as a product of multiple Mellin-Barnes contour integral by using (6) and interchanging the order of integration, which is permissible under the above stated conditions and appeal to the Euler's integral of the first kind [3, p. 1]. After little arrangement we finally arrive at (12). ■

Corollary 2.3. *When $n_i = 0, p_i = 0, q_i = 0, i = 2, \cdots, r-1$ (the empty product denotes unity) the* (12) *reduces to the multiple Laplace transformation of multivariable H-function of r variables.*

Corollary 2.4. *When $r = 2$ and $n_2 = 0, p_2 = 0, q_2 = 0$ in* (12)*, it reduces to the double Laplace transformation of H-function of two variables* [2, p. 148].

Theorem 2.3 Suppose the conditions (9) to be satisfied. Then Multiple Laplace transform of multivariable I-function as follows

$$(\mathfrak{L}I)(s) = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^r s_i t_i\right) \prod_{i=1}^r t_i^{\rho_i - 1} I[z_1 t_1^{-\mu_1}, \cdots, z_r t_r^{-\mu_r}] dt_1 \cdots dt_r$$

$$= \prod_{i=1}^r \left\{s_i^{-\rho_i}\right\} I^{\{0,n_i\}_{2,r}:\left\{(m^{(i)}+1,n^{(i)})\right\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\left\{(p^{(i)},q^{(i)}+1)\right\}^{1,r}} \left[\begin{array}{c|c} z_1 s_1^{\mu_1} & \mathcal{A} : \mathcal{B} \\ \vdots & \mathcal{C} : \mathcal{D} \\ z_r s_r^{\mu_r} & \end{array}\right] \tag{13}$$

where $\mu_i > 0, \Re(s_i) > 0, \Re(\rho_i) - \mu_i \max_{1 \le j \le n^{(i)}} \Re\left(\frac{a_j^{(i)} - 1}{\alpha_j^{(i)}}\right) > 0, i = 1, \cdots, r,$

$$\mathcal{A} \equiv \left\{(a_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)})_{1,p_i}^{2,r}\right\},\ \mathcal{B} \equiv \left\{\left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}\right\},$$

$$\mathcal{C} \equiv \left\{(b_{rj}; \beta_{rj}^{(1)}, \cdots, \beta_{rj}^{(r)})_{1,q_i}^{2,r}\right\},\ \mathcal{D} \equiv \left\{(\rho_i, \mu_i), \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}\right\}$$

Proof The proof of (13) is same as (12). ■

Corollary 2.5. *When $n_i = 0$, $p_i = 0$, $q_i = 0$, $i = 2, \cdots, r-1$ (the empty product denotes unity) the* (13) *reduces to the multiple Laplace transformation of multivariable H-function of r variables.*

Corollary 2.6. *When $r = 2$ and $n_2 = 0$, $p_2 = 0$, $q_2 = 0$ in* (13)*, it reduces to the double Laplace transformation of H-function of two variables* [2, p. 148].

Theorem 2.4 Suppose the conditions (9) to be satisfied and $\mu_i > 0, \nu_i > 0, n_i = 0, n_i' = 0$, $i = 1, \cdots, r$. Then the Multiple Mellin transform of the product of two I-function of r variable defined in (6) as follows

$$(\mathfrak{M}I \cdot I)(s) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r t_i^{s_i - 1} I[z_1 t_1^{\mu_1}, \cdots, z_r t_r^{\mu_r}] I'[\eta_1 t_1^{\nu_1}, \cdots, \eta_r t_r^{\nu_r}] dt_1 \cdots dt_r$$

$$= \prod_{i=1}^r \left\{\frac{\eta_i^{-\frac{s_i}{\nu_i}}}{\nu_i}\right\} I^{\{0,0\}_{2,r}:\left\{(m^{(i)}+n'^{(i)},m'^{(i)}+n^{(i)})\right\}^{1,r}}_{\left\{p_i+q_i',p_i'+q_i\right\}_{2,r}:\left\{(p^{(i)}+q'^{(i)},p'^{(i)}+q^{(i)})\right\}^{1,r}} \left[\begin{array}{c|c} z_1 \eta_1^{-\frac{s_1}{\nu_1}} & \mathcal{A}:\mathcal{B} \\ \vdots & \mathcal{C}:\mathcal{D} \\ z_r \eta_r^{-\frac{s_r}{\nu_r}} & \end{array}\right] \tag{14}$$

where

$$-\mu_i \min_{1\le j\le m^{(i)}} \Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) - \nu_i \min_{1\le j\le m'^{(i)}} \Re\left(\frac{b_j^{'(i)}}{\beta_j^{'(i)}}\right) < \Re(s_i) <$$
$$\mu_i \min_{1\le j\le n^{(i)}} \Re\left(\frac{1-a_j^{(i)}}{\alpha_j^{(i)}}\right) + \nu_i \min_{1\le j\le n'^{(i)}} \Re\left(\frac{1-a_j^{'(i)}}{\alpha_j^{'(i)}}\right),$$

$i = 1, \cdots, r$ and

$$\mathcal{A} \equiv \left\{\left(a_{ij}; \alpha_{ij}^{(1)}, \cdots, \alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}; \left(1 - b'_{ij} - \sum_{k=1}^{i} \frac{s_k}{\nu_k}\beta_{ij}^{'(k)}; \frac{\mu_1}{\nu_1}\beta_{ij}^{'(1)}, \cdots, \frac{\mu_i}{\nu_i}\beta_{ij}^{'(i)}\right)_{1,q'_i}^{2,r}\right\},$$
$$\mathcal{B} \equiv \left\{\left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}, \left(1 - b_j^{'(i)} - \frac{s_i}{\nu_i}\beta_j^{'(i)}, \frac{\mu_i}{\nu_i}\beta_j^{'(i)}\right)_{1,q'^{(i)}}^{1,r}\right\},$$
$$\mathcal{C} \equiv \left\{\left(b_{ij}; \beta_{ij}^{(1)}, \cdots, \beta_{ij}^{(i)}\right)_{1,q_i}^{2,r}; \left(1 - a'_{ij} - \sum_{k=1}^{i} \frac{s_k}{\nu_k}\alpha_{ij}^{'(k)}; \frac{\mu_1}{\nu_1}\alpha_{ij}^{'(1)}, \cdots, \frac{\mu_i}{\nu_i}\alpha_{ij}^{'(i)}\right)_{1,p'_i}^{2,r}\right\},$$
$$\mathcal{D} \equiv \left\{\left(b_j^{(i)}, \beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}; \left(1 - a_j^{'(i)} - \frac{s_i}{\nu_i}\alpha_j^{'(i)}, \frac{\mu_i}{\nu_i}\alpha_j^{'(i)}\right)_{1,p'^{(i)}}^{1,r}\right\}$$

Proof To prove the integral formula (14), we express the first I-function on its left-hand-side as a multiple Mellin-Barnes contour integral by using (6) and interchange the order of integration, which is permissible under the above stated conditions. Evaluate the inner integral with the help of (11), after straight calculation we finally arrive at (14). ■

Corollary 2.7. *When $p_i = 0$, $q_i = 0$, $p'_i = 0$, $q'_i = 0$, $i = 2, \cdots, r-1$ the empty product denotes unity) the* (14) *reduces to the multiple Mellin transformation of H-function of r variables.*

Corollary 2.8. *When $r = 2$ and $\nu_1 = 1$, $\nu_2 = 1$, $p_2 = 0$, $q_2 = 0$, $p_2^{'} = 0$, $q_2^{'} = 0$ in* (14)*, it reduces to the double Mellin transformation of H-function of two variables* [2, p. 148].

Theorem 2.5 Suppose the conditions (9) to be satisfied and $\mu_i > 0$, $\nu_i > 0$, $n_i = 0, n'_i = 0$, $i \in \{1, \cdots, r\}$. Then the Multiple Mellin transform of the product of two I-function of r variable define in (6) as follows

$$(\mathfrak{M}I \cdot I)(s) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} t_i^{s_i-1} I[z_1 t_1^{-\mu_1}, \cdots, z_r t_r^{-\mu_r}] I'[\eta_1 t_1^{\nu_1}, \cdots, \eta_r t_r^{\nu_r}] dt_1 \cdots dt_r$$
$$= \prod_{i=1}^{r}\left\{\frac{\eta_i^{-\frac{s_i}{\nu_i}}}{\nu_i}\right\} I^{\{0,0\}_{2,r}:\left\{(m^{(i)}+m'^{(i)}, n^{(i)}+n'^{(i)})\right\}^{1,r}}_{\left\{p_i+p'_i, q_i+q'_i\right\}_{2,r}:\left\{(p^{(i)}+p'^{(i)}, q^{(i)}+q'^{(i)})\right\}^{1,r}} \left[\begin{matrix} z_1\eta_1^{\frac{s_1}{\nu_1}} \\ \vdots \\ z_r\eta_r^{\frac{s_r}{\nu_r}} \end{matrix} \middle| \begin{matrix} \mathcal{A}:\mathcal{B} \\ \mathcal{C}:\mathcal{D} \end{matrix}\right] \tag{15}$$

where

$$-\mu_i \min_{1\le j\le n^{(i)}} \Re\left(\frac{1-a_j^{(i)}}{\alpha_j^{(i)}}\right) - \nu_i \min_{1\le j\le m'^{(i)}} \Re\left(\frac{b_j^{'(i)}}{\beta_j^{'(i)}}\right) < \Re(s_i) <$$

$$\mu_i \min_{1\le j\le m^{(i)}} \Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) + \nu_i \min_{1\le j\le n'^{(i)}} \Re\left(\frac{1-a_j^{'(i)}}{\alpha_j^{'(i)}}\right),$$

$i = 1, \cdots, r$ and

$$\mathcal{A} \equiv \left\{\left(a_{ij}; \alpha_{ij}^{(1)}, \cdots, \alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}; \left(a'_{ij} + \sum_{k=1}^{i} \frac{s_k}{\nu_k}\alpha_{ij}^{'(k)}; \frac{\mu_1}{\nu_1}\alpha_{ij}^{'(1)}, \cdots, \frac{\mu_i}{\nu_i}\alpha_{ij}^{'(i)}\right)_{1,p'_i}^{2,r}\right\},$$

$$\mathcal{B} \equiv \left\{\left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}; \left(a_j^{'(i)} + \frac{s_i}{\nu_i}\alpha_j^{'(i)}, \frac{\mu_i}{\nu_i}\alpha_j^{'(i)}\right)_{1,p'^{(i)}}^{1,r}\right\},$$

$$\mathcal{C} \equiv \left\{\left(b_{ij}; \beta_{ij}^{(1)}, \cdots, \beta_{ij}^{(i)}\right)_{1,q_i}^{2,r}; \left(b'_{ij} + \sum_{k=1}^{i} \frac{s_k}{\nu_k}\beta_{ij}^{'(k)}; \frac{\mu_1}{\nu_1}\beta_{ij}^{'(1)}, \cdots, \frac{\mu_i}{\nu_i}\beta_{ij}^{'(i)}\right)_{1,q'_i}^{2,r}\right\},$$

$$\mathcal{D} \equiv \left\{\left(b_j^{(i)}, \beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}; \left(b_j^{'(i)} + \frac{s_i}{\nu_i}\beta_j^{'(i)}, \frac{\mu_i}{\nu_i}\beta_j^{'(i)}\right)_{1,q'^{(i)}}^{1,r}\right\}$$

Proof To prove the integral formula (15) same as (14). ■

Corollary 2.9. *When $n_i = 0, p_i = 0, q_i = 0, n'_i = 0, p'_i = 0, q'_i = 0$, $i = 2, \cdots, r-1$ and $n_r = 0, n'_r = 0$ (the empty product denotes unity) the* (15) *reduces to the multiple Mellin transformation of H-function of r variables.*

Corollary 2.10. *When $n_i = 0, p_i = 0, q_i = 0, n'_i = 0, p'_i = 0, q'_i = 0$, $i = 2, \cdots, r-1$, $n_r = 0, n'_r = 0$ and $\nu_k = 1, k = 1, \cdots, r$ then the* (15) *reduces to the result of* [11].

Corollary 2.11. *When $r = 2$ and $\nu_1 = 1$, $\nu_2 = 1$, $p_2 = 0$, $q_2 = 0$, $p_2^{'} = 0$, $q_2^{'} = 0$ in* (15)*, it reduces to the double Mellin transformation of H-function of two variables* [2, p. 149].

3. Conclusion

The I-function of r variables defined by Prasad [17] and Prasad and Yadav [18] in terms of the Mellin-Barnes type of basic integrals is most general in character which involves a number of special functions. The results deduced in the present paper may provide better multiple Mellin and Laplace transforms of some simpler multivariable special functions.

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B.K. Dutta
Department of Mathematics, NERIST, Nirjuli, Arunachal Pradesh, India
E-mail address: dutta.bk11@gmail.com

L.K. Arora
Department of Mathematics, NERIST, Nirjuli, Arunachal Pradesh, India
E-mail address: lkarora_nerist@yahoo.com