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EXISTENCE OF POSITIVE CONTINUOUS SOLUTION OF A QUADRATIC INTEGRAL EQUATION OF FRACTIONAL ORDERS

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Abstract. We are concerned here with the existence of at least one positive continuous solution of the quadratic integral equation of fractional orders

$$x(t) \ = \ a(t) + I^{\alpha} f(t, x(t)) \cdot I^{\beta} g(t, x(t)), \ \alpha, \ \beta \in (0, 1].$$

The maximal and minimal solutions are also proved. Some applications are given.

1. Introduction

Quadratic integral equations (QIEs) are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equations can be very often encountered in many applications.
The quadratic integral equations have been studied in several papers and monographs (see for examples [1]-[8] and [10]-[18]).

The quadratic integral equation

$$x(t) \ = \ a(t) \ + \ \int_0^t \ f(s, x(s)) \ ds \cdot \int_0^t g(s, x(s)) \ ds, \ t \in [0, T] \tag{1}$$

has been studied in [12]. The authors proved that it has at least one continuous solution, also they proved the existence of the maximal and minimal solutions.

Let $\alpha, \ \beta \in (0, 1]$. Here we are concerned with the quadratic integral equation of fractional orders

$$x(t) \ = \ a(t) + I^{\alpha} f(t, x(t)) \cdot I^{\beta} g(t, x(t)), \ t \in [0, T]. \tag{2}$$

We prove the existence of positive continuous solution of (2).

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The proof of the main result will be based on the following fixed-point theorem.
Theorem 1.1 Tychonov fixed-point Theorem [9]
Suppose B is a complete, locally convex linear space and S is a closed convex subset of B. Let the mapping $T: B \to B$ be continuous and $T(S) \subset S$. If the closure of $T(S)$ is compact, then T has a fixed point in S.

Let $I = [0,T]$, $L_1 = L_1[0,T]$ be the space of all Lebesgue integrable functions on I.

Now, the definition of the fractional-order integral operator is given by.
Definition 1.1 Let β be a positive real number, the fractional-order integral of order β of the function f is defined on the interval $[a,b]$ by (see [19] and [20])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds,$$

and when $a = 0$, we have $I^\beta f(t) = I_0^\beta f(t)$.

For further properties of fractional-order integral operator see [19]-[20] for example.

2. Main results

Consider the quadratic integral equation (2), under the following assumptions

(i) $a(t): I = [0,T] \to R_+$ is continuous;
(ii) $f, g: I \times R_+ \to R_+$ satisfy Carathèodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for almost all $t \in [0,T]$) and there exist two functions $m_1,\ m_2 \in L_1$ such that

$$f(t,x) \leq m_1(t),\ g(t,x) \leq m_2(t)\ \forall\ (t,x) \in I \times R_+.$$

Now, we have the following theorem.
Theorem 2.1 Let the assumptions (i) and (ii) are satisfied, then the quadratic integral equation (2) has at least one positive solution $x \in C(I)$.
Proof. Let $C = C(I)$ be the space of all continuous functions on $[0,T]$. It can be verified that $C(I)$ is a complete locally convex linear space [9].
Define subset S of $C(I)$ by

$$S = \{\ x \in C:\ 0 < x(t) \leq r\ \},\ t \in I.$$

Let $\gamma_i <$ Max $\{\alpha,\ \beta\}$, $i = 1,2$. Then we can write

$$x(t) = a(t) + I^{\alpha-\gamma_1} I^{\gamma_1}\ f(t,x(t)) \cdot I^{\beta-\gamma_2} I^{\gamma_2}\ g(t,x(t))$$

and

$$|\ x(t)\ | \leq a(t) + I^{\alpha-\gamma_1} |\ I^{\gamma_1}\ f(t,x(t))| \cdot I^{\beta-\gamma_2} |\ I^{\gamma_2}\ g(t,x(t))\ |.$$

Let $M_i = \max\{I^{\gamma_i} m_i(t):\ t \in [0,T],\ \gamma_i < \alpha\ \}$, $i = 1,2$.

Then from assumptions (i) and (ii) we can get

$$\begin{aligned}
|\ x(t)\ | &\leq a(t) + I^{\alpha-\gamma_1} |\ I^{\gamma_1}\ m_1(t)| \cdot I^{\beta-\gamma_2} |\ I^{\gamma_2}\ m_2(t)\ | \\
&\leq K + M_1 \int_0^t \frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\, ds \cdot M_2 \int_0^t \frac{(t-s)^{\beta-\gamma_2-1}}{\Gamma(\beta-\gamma_2)}\, ds \\
&\leq K + \frac{M_1 T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} \cdot \frac{M_2 T^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}
\end{aligned}$$

where $K = \sup_{t \in I} a(t)$.

From the last estimate we deduce that $r = K + \frac{M_1 T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} \cdot \frac{M_2 T^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}$. It is clear that the set S is closed and convex.

Define the operator $H : S \to C(I)$ by

$$Hx(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \cdot \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds. \tag{3}$$

Assumption (ii) implies that H is a continuous operator in x. We shall prove that $HS \subset S$.

For every $x \in S$ we have

$$| Hx(t) | \le | a(t) | + I^{\alpha-\gamma_1} | I^{\gamma_1} f(t,x(t))| \cdot I^{\beta-\gamma_2} | I^{\gamma_2} g(t,x(t)) |,$$

and

$$| Hx(t) | \le K + \frac{M_1 T^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)} \cdot \frac{M_2 T^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}.$$

Then, $Hx \in S$ and hence $HS \subset S$.

Now for t_1 and $t_2 \in [0,T]$ (without loss of generality assume that $t_1 < t_2$), we have

$$\begin{aligned}
Hx(t_2) - Hx(t_1) &= a(t_2) - a(t_1) \\
&+ \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \cdot \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds \\
&- \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \cdot \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds \\
&= a(t_2) - a(t_1) \\
&+ \left(\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \right) \\
&\cdot \left(\int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds \right) \\
&- \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \cdot \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds \\
&\le a(t_2) - a(t_1) + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \cdot \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \cdot \int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\, ds \cdot \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s))\, ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
| Hx(t_2) - Hx(t_1) | &\le | a(t_2) - a(t_1) | + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_1(s)ds \cdot \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} m_2(s)ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} m_1(s)ds \cdot \int_0^{t_1} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} m_2(s)ds
\end{aligned}$$

$$+\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\ m_1(s)ds \cdot \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\ m_2(s)ds$$
$$\leq \mid a(t_2)-a(t_1) \mid + M_1 \cdot M_2 \left[\frac{T^{\alpha-\gamma_1}-(t_2-t_1)^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}\right]\left[\frac{(t_2-t_1)^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}\right]$$
$$+M_1 \cdot M_2 \left[\frac{(t_2-t_1)^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}\right]\left[\frac{T^{\beta-\gamma_2}-(t_2-t_1)^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}\right]$$
$$+M_1 \cdot M_2 \left[\frac{(t_2-t_1)^{\alpha-\gamma_1}}{\Gamma(\alpha-\gamma_1+1)}\right]\left[\frac{(t_2-t_1)^{\beta-\gamma_2}}{\Gamma(\beta-\gamma_2+1)}\right].$$

Hence
$$\mid t_2-t_1 \mid < \delta \Longrightarrow \ \mid x_n(t_2)-x_n(t_1) \mid < \varepsilon(\delta).$$

This means that the functions of HS are equi-continuous on $[0,T]$.

Then by the Arzela-Ascoli Theorem [9] the closure of HS is compact . Since all conditions of the Tychonov Fixed-point Theorem hold, then H has a fixed point in S. Consequently, the quadratic integral equation (2) has a positive continuous solution in S. ■

Letting $\beta,\ \alpha,\ \gamma_1$ and $\gamma_2 \to 1$, then we have the following corollary (which is the same results obtained in [12]).

Corollary 2.1 Let the assumptions (i) and (ii) be satisfied, then the quadratic integral (1) has at least one continuous solution.

When $f = g$ and $\alpha = \beta$ we have the following corollary

Corollary 2.2 Let the assumptions (i) and (ii) be satisfied. Then he quadratic integral equation
$$x(t) = (I^{\alpha} f(t, x(t)))^2 \tag{4}$$
has at least one positive continuous solution.

3. Maximal and minimal solutions

Definition 3.1 Let $q(t)$ be a solution of the quadratic integral equation (2). Then $q(t)$ is said to be a maximal solution of (2) if every solution of it satisfies the inequality
$$x(t) < q(t)\ t \in I. \tag{5}$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality.

Lemma 3.1 Let $f(t,x),\ g(t,x)\ \in\ L_1$ and $u(t),\ v(t)$ be two continuous functions on $[0,T]$ satisfying
$$u(t) \leq a(t) + I^{\alpha} f(t,u(t)) \cdot I^{\beta} g(t,u(t))$$
$$v(t) \geq a(t) + I^{\alpha} f(t,v(t)) \cdot I^{\beta} g(t,v(t))$$
and one of them is strict. If $f,\ g$ are monotonic nondecreasing in x, then
$$u(t) < v(t),\ t > 0 \tag{6}$$

Proof. Let the conclusion (6) be false, then there exists t_1 such that
$$u(t_1) = v(t_1),\ t_1 > 0$$

and

$$u(t) < v(t),\ 0 < t < t_1$$

From the monotonicity of $f,\ g$ in x, we get

$$\begin{aligned} u(t_1) &\leq a(t_1) + I^{\alpha} f(t_1, u(t_1)) \cdot I^{\beta} g(t_1, u(t_1)) \\ &= a(t_1) + \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds \cdot \int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)} g(s, u(s))ds \\ &< a(t_1) + \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s))ds \cdot \int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)} g(s, v(s))ds < v(t_1) \end{aligned}$$

which contradicts the fact that $u(t_1) = v(t_1)$, then $u(t) < v(t)$.

Now, for the existence of the maximal and minimal solutions we have the following theorem.

Theorem 3.1 Let $f(t, x)$ and $g(t, x)$ satisfy the assumptions (i) and (ii), suppose that $f(t, x)$ and $g(t, x)$ are monotonic nondecreasing in x for each $t \in I$, then there exist maximal and minimal solutions for the quadratic integral equation (2).

Proof. Firstly, we prove the existence of the maximal solution of (2). Let $\epsilon > 0$ and consider the quadratic integral equation

$$x_{\epsilon}(t) = a(t) + I^{\alpha} f_{\epsilon}(t, x_{\epsilon}(t)) \cdot I^{\beta} g_{\epsilon}(t, x_{\epsilon}(t)) \tag{7}$$

where

$$\begin{aligned} f_{\epsilon}(t, x_{\epsilon}(t)) &= f(t, x_{\epsilon}(t)) + \epsilon; \\ g_{\epsilon}(t, x_{\epsilon}(t)) &= g(t, x_{\epsilon}(t)) + \epsilon. \end{aligned}$$

Clearly, the functions $f_{\epsilon}(t, x_{\epsilon}(t)),\ g_{\epsilon}(t, x_{\epsilon}(t))$ satisfy Carathèodory condition and

$$\begin{aligned} |\ f_{\epsilon}(t, x_{\epsilon}(t))\ | &\leq m_1(t) + \epsilon \\ |\ g_{\epsilon}(t, x_{\epsilon}(t))\ | &\leq m_2(t) + \epsilon \end{aligned}$$

and

$$\begin{aligned} I^{\gamma_1} m_1(t) + I^{\gamma_1} \epsilon &\leq M_1\ \forall\ \gamma_1 < \alpha \\ I^{\gamma_2} m_2(t) + I^{\gamma_2} \epsilon &\leq M_2\ \forall\ \gamma_2 < \beta. \end{aligned}$$

Therefore (7) has a continuous solution $x_{\epsilon}(t)$. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$\begin{aligned} x_{\epsilon_2}(t) &= a(t) + I^{\alpha} f_{\epsilon_2}(t, x_{\epsilon_2}(t)) \cdot I^{\beta} g_{\epsilon_2}(t, x_{\epsilon_2}(t)) \\ &= a(t) + (I^{\alpha} f(t, x_{\epsilon_2}(t)) + I^{\alpha}\epsilon_2) \cdot (I^{\beta} g(t, x_{\epsilon_2}(t)) + I^{\beta}\epsilon_2) \quad (8) \\ x_{\epsilon_1}(t) &= a(t) + I^{\alpha} f_{\epsilon_1}(t, x_{\epsilon_1}(t)) \cdot I^{\beta} g_{\epsilon_1}(t, x_{\epsilon_1}(t)) \\ &= a(t) + (I^{\alpha} f(t, x_{\epsilon_1}(t)) + I^{\alpha}\epsilon_1) \cdot (I^{\beta} g(t, x_{\epsilon_1}(t)) + I^{\beta}\epsilon_1) \quad (9) \\ x_{\epsilon_1}(t) &> a(t) + (I^{\alpha} f(t, x_{\epsilon_1}(t)) + I^{\alpha}\epsilon_2) \cdot (I^{\beta} g(t, x_{\epsilon_1}(t)) + I^{\beta}\epsilon_2) \end{aligned}$$

Applying the above Lemma on (8) and (9), we have $x_{\epsilon_2}(t) < x_{\epsilon_1}(t)$ for $t \in [0, T]$ as shown before, the family of functions $x_{\epsilon}(t)$ is equi-continuous and uniformly bounded.

Hence, by Arzela-Ascoli Theorem, there exists a decreasing sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ an $n \to \infty$, and $\lim\limits_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and denote this limit by $q(t)$.

From the continuity of the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ in the second argument, we get

$$f_{\epsilon_n}(t, x_{\epsilon_n}(t)) \to f(t, x(t)) \ as \ n \to \infty$$
$$g_{\epsilon_n}(t, x_{\epsilon_n}(t)) \to g(t, x(t)) \ as \ n \to \infty$$

and

$$q(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = a(t) + I^\alpha f(t, q(t)) \cdot I^\beta g(t, q(t))$$

yields $q(t)$ is a solution of (7). Finally, we shall show that $q(t)$ is the maximal solution of (7).

To do this, let $x(t)$ be any solution of (7), then

$$\begin{aligned} x_\epsilon(t) &= a(t) + (I^\alpha f(t, x_\epsilon(t)) + I^\alpha \epsilon) \cdot (I^\beta g(t, x_\epsilon(t)) + I^\beta \epsilon) \\ &> a(t) + I^\alpha f(t, x_\epsilon(t)) \cdot I^\beta g(t, x_\epsilon(t)) \end{aligned}$$

and

$$x(t) = a(t) + I^\alpha f(t, x(t)) \cdot I^\beta g(t, x(t)).$$

Applying the above Lemma, then we have $x_\epsilon(t) > x(t)$ for $t \in [0, T]$. From the uniqueness of the maximal solution, it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in [0, T]$ as $\epsilon \to 0$. By a similar way as done above, we set

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) - \epsilon$$
$$g_\epsilon(t, x_\epsilon(t)) = g(t, x_\epsilon(t)) - \epsilon$$

and prove the existence of minimal solution. ■

4. Fractional order differential equation

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order.

Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention.

For the fractional-order differential equation

$$_*D^\alpha \sqrt{x(t)} = f(t, x(t)), \ t > 0 \tag{10}$$

with the initial condition

$$I^{1-\alpha} \sqrt{x(t)}|_{t=0} = 0 \tag{11}$$

we can prove the following corollaries.

Corollary 4.1 The initial value problem (10) and (11) is equivalent to the quadratic integral equation

$$x(t) = (I^\alpha f(t, x(t)))^2 \tag{12}$$

Corollary 4.2 The initial value problem (10) and (11) has at least one solution $x \in C(I)$.

Corollary 4.2 If $f(t, x)$ is monotonic nondecreasing in x for each $t \in [0, T]$, then there exist maximal solution and minimal solutions of the problem (10) and (11).

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