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MULTIVARIATE LANDAU FRACTIONAL INEQUALITIES

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Abstract. Here we establish multivariate left Caputo fractional L_p-Landau type inequalities, $p \in (1, \infty]$. We give applications on $\mathbb{R}^N$, $N \geq 1$.

1. Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$||f'||_{p,I} \leq C_p(I) ||f||_{p,I}^{\frac{1}{2}} ||f''||_{p,I}^{\frac{1}{2}} \tag{1}$$

where $||.||_{p,I}$ is the p-norm on the interval I, see [1], [5].

The research on these inequalities started by E. Landau [11] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \quad \text{and} \quad C_\infty(\mathbb{R}) = \sqrt{2}, \tag{2}$$

are the best constants in (1).

In 1932, G. H. Hardy and J.E. Littlewood [8] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G. H. Hardy, E. Landau and J.E. Littlewood [9] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \quad \text{for} \quad p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [7] and [10], was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

In this article we prove multivariate fractional Landau inequalities with respect to $||.||_p$, $p \in (1, \infty]$, involving the left Caputo fractional radial derivative.

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2. Main results

We need

Definition 1. ([6], p.38) Let $f \in AC^m([a,b])$, $m \in \mathbb{N}$ (i.e. $f^{(m-1)} \in AC([a,b])$),where $m = \lceil \alpha \rceil$, $\alpha > 0$, ($\lceil . \rceil$ the ceiling of the number). We define the left Caputo fractional derivative of order $\alpha > 0$, by

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad (5)$$

$\forall\, x \in [a,b]$. *We set* $D_{*a}^{0} f(x) = f(x)$, $\forall x \in [a,b]$.

Here $D_{*a}^{\alpha} f$ exists a.e. on $[a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b])$, see [6], pp.13 and 37-38. When $\alpha = m \in \mathbb{N}$, then

$$D_{*a}^{m} f(x) = f^{(m)}(x),\ \forall x \in [a,b].$$

If $x < a$ we define $D_{*a}^{\alpha} f(x) = 0$.

We make

Remark 2. Here we follow [12], p. 149-150.

For $x \in \mathbb{R}^N - \{0\}, N > 1$, we can write uniquely $x = rw$, $r = |x| > 0$, $w = x/r \in S^{N-1}$, $|w| = 1$. Clearly

$$\mathbb{R}^N - \{0\} = (0,\infty) \times S^{N-1}, \quad \text{where} \quad S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}.$$

Let $A > 0$, we consider

$$B(0,A) := \{x \in \mathbb{R}^N : |x| < A\} \subseteq \mathbb{R}^N, \quad |x| - \text{ Euclidean norm of } x \in \mathbb{R}^N.$$

We consider

$$\mathbb{R}^N - B(0,A) = [A, +\infty) \times S^{N-1}, \quad (6)$$

on which we establish Landau fractional inequalities. We need to define the left Caputo radial fractional derivative for our case, see also [2], p.421.

Definition 3. Let $f : \mathbb{R}^N - B(0,A) \to \mathbb{R}$, $\nu \geq 0$, $m := \lceil \nu \rceil$, such that $f(\cdot w) \in AC^m([A,b])$, $\forall b > A$, for all $w \in S^{N-1}$. We call the left Caputo radial fractional derivative the following function

$$\frac{\partial_{*A}^{\nu} f(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(m-\nu)} \int_A^r (r-t)^{m-\nu-1} \frac{\partial^m f(tw)}{\partial r^m} dt, \quad (7)$$

where $x \in \mathbb{R}^N - B(0,A)$; that is $x = rw, r \in [A,\infty), w \in S^{N-1}$.

Clearly

$$\frac{\partial_{*A}^{0} f(x)}{\partial r^{0}} = f(x), \quad (8)$$

$$\frac{\partial_{*A}^{\nu} f(x)}{\partial r^{\nu}} = \frac{\partial^{\nu} f(x)}{\partial r^{\nu}}, \quad \text{if} \quad \nu \in \mathbb{N}. \quad (9)$$

The above function (7) exists almost everywhere for $x \in \mathbb{R}^N - B(0,A)$, see [2], p. 422.

We make

Remark 4. Let $0 < \nu \leq 1$; $A > 0$ be fixed, with $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $\forall\, w \in S^{N-1}$. Assume $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$,

$$\left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{\infty,\mathbb{R}^N - B(0,A)} < \infty$$

and

$$||D_{*a}^{\nu+1}f(\cdot w)||_{\infty,[a,+\infty)} \leq ||D_{*A}^{\nu+1}f(\cdot w)||_{\infty,[A,+\infty)},\ \forall\, a \geq A, \forall\, w \in S^{N-1}. \tag{10}$$

Then, by Theorem 8 of [3], we get that

$$||f'(\cdot w)||_{\infty,[A,+\infty)} \leq (\nu+1)\left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} \cdot \Big(\Gamma(\nu+2)\Big)^{-\frac{1}{\nu+1}} \cdot$$

$$\cdot \left(||f(\cdot w)||_{\infty,[A,+\infty)}\right)^{\frac{\nu}{\nu+1}} \cdot \left(||D_{*A}^{\nu+1}f(\cdot w)||_{\infty,[A,+\infty)}\right)^{\frac{1}{\nu+1}} \leq \tag{11}$$

$$(\nu+1)\left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} \cdot \Big(\Gamma(\nu+2)\Big)^{-\frac{1}{\nu+1}} \cdot \left(||f||_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\frac{\nu}{\nu+1}} \cdot \left(\left\|\frac{\partial_{*A}^{\nu+1}f}{\partial r^{\nu+1}}\right\|_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\frac{1}{\nu+1}} =: \theta.$$

Hence

$$\left\|\frac{\partial f}{\partial r}\right\|_{\infty,\mathbb{R}^N-B(0,A)} \leq \theta. \tag{12}$$

We have proved the multivariate fractional Landau inequality,

Theorem 5. Let $0 < \nu \leq 1$; $A > 0$ be fixed, with $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $\forall\, w \in S^{N-1}$. Assume $||f||_{\infty,\mathbb{R}^N-B(0,A)} < \infty$,

$$\left\|\frac{\partial_{*A}^{\nu+1}f}{\partial r^{\nu+1}}\right\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty$$

and

$$||D_{*a}^{\nu+1}f(\cdot w)||_{\infty,[a,+\infty)} \leq ||D_{*A}^{\nu+1}f(\cdot w)||_{\infty,[A,+\infty)},\ \forall\, a \geq A, \forall\, w \in S^{N-1}. \tag{13}$$

Then

$$\left\|\frac{\partial f}{\partial r}\right\|_{\infty,\mathbb{R}^N-B(0,A)} \leq (\nu+1)\left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} \cdot \Big(\Gamma(\nu+2)\Big)^{-\frac{1}{\nu+1}} \cdot$$

$$\cdot \left(||f||_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\frac{\nu}{\nu+1}} \cdot \left(\left\|\frac{\partial_{*A}^{\nu+1}f}{\partial r^{\nu+1}}\right\|_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\frac{1}{\nu+1}}. \tag{14}$$

We give when $\nu = 1$,

Corollary 6. Let $A > 0$ fixed, with $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $\forall\, w \in S^{N-1}$. Assume $||f||_{\infty,\mathbb{R}^N-B(0,A)} < \infty$,

$$\left\|\frac{\partial^2 f}{\partial r^2}\right\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty.$$

Then

$$\left\|\frac{\partial f}{\partial r}\right\|_{\infty,\mathbb{R}^N-B(0,A)} \leq 2 \cdot \sqrt{||f||_{\infty,\mathbb{R}^N-B(0,A)} \cdot \left\|\frac{\partial^2 f}{\partial r^2}\right\|_{\infty,\mathbb{R}^N-B(0,A)}}\ . \tag{15}$$

We make

Remark 7. All entities here are assumed to make sense and to be well-defined. We see that $(q > 1)$:

$$\left\|D_{*A}^{\nu+1}f(\cdot w)\right\|_{q,[A,+\infty)}^q = \int_A^{+\infty} \left|D_{*A}^{\nu+1}f(rw)\right|^q dr =$$

$$\int_A^{+\infty} \left|D_{*A}^{\nu+1}f(rw)\right|^q \cdot r^{N-1} \cdot r^{1-N} dr \tag{16}$$

(see $r \geq A$ implies $r^{1-A} \leq A^{1-N}$, $N > 1$).

Hence it holds

$$\left\|D_{*A}^{\nu+1} f(\cdot w)\right\|_{q,[A,+\infty)}^{q} \leq A^{1-N} \cdot \int_{A}^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^{q} \cdot r^{N-1} dr. \tag{17}$$

Notice that

$$\frac{\int\limits_{S^{N-1}} dw}{w_N} = 1, \quad \text{where} \quad w_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{18}$$

Therefore we get

$$\left\|D_{*A}^{\nu+1} f(\cdot w)\right\|_{q,[A,+\infty)}^{\frac{1}{\left(\nu+\frac{1}{p}\right)}} \leq \tag{19}$$

$$A^{\left(\frac{1-N}{q}\right) \cdot \frac{1}{\left(\nu+\frac{1}{p}\right)}} \cdot \left(\int_{A}^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^{q} \cdot r^{N-1} dr\right)^{\frac{1}{q} \cdot \left(\frac{1}{\nu+\frac{1}{p}}\right)}.$$

Set

$$c_\nu := \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \frac{1}{\left(\Gamma(\nu)\right)^{\frac{1}{\left(\nu+\frac{1}{p}\right)}} \cdot \left(p(\nu-1)+1\right)^{\frac{1}{p\nu+1}}}. \tag{20}$$

So here we have $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $1 - \frac{1}{p} < \nu \leq 1$, $f(\cdot w) \in AC^2([A,b])$, $\forall b > A$, $\forall w \in S^{N-1}$, $A > 0$ is fixed.

Assume $D_{*A}^{\nu+1} f(\cdot w) \in L_q([A,+\infty))$, $\forall w \in S^{N-1}$, $||f||_{\infty,\mathbb{R}^N - B(0,A)} < \infty$ and

$$\left\|D_{*a}^{\nu+1} f(\cdot w)\right\|_{q,[a,+\infty)} \leq \left\|D_{*A}^{\nu+1} f(\cdot w)\right\|_{q,[A,+\infty)}, \forall\, a \geq A,\ \forall\, w \in S^{N-1}. \tag{21}$$

Hence by Theorem 8 of [4] and (19), we get

$$||f'(\cdot w)||_{\infty,[A,+\infty)} \leq c_\nu \cdot A^{\left(\frac{1-N}{q}\right) \cdot \frac{1}{\left(\nu+\frac{1}{p}\right)}} \cdot$$

$$\cdot \left(||f(\cdot w)||_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\int_{A}^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^{q} \cdot r^{N-1} dr\right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}. \tag{22}$$

We call

$$\beta_\nu := c_\nu \cdot A^{\left(\frac{1-N}{q}\right) \cdot \frac{1}{\left(\nu+\frac{1}{p}\right)}}, \tag{23}$$

and

$$\gamma_\nu := \beta_\nu \cdot \left(||f||_{\infty,\mathbb{R}^N - B(0,A)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)}. \tag{24}$$

Therefore

$$||f'(\cdot w)||_{\infty,[A,+\infty)} \leq \tag{25}$$

$$\beta_\nu \cdot \left(||f(\cdot w)||_{\infty,[A,+\infty)}\right)^{\left(\frac{\nu-1+\frac{1}{p}}{\nu+\frac{1}{p}}\right)} \cdot \left(\int_{A}^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^{q} \cdot r^{N-1} dr\right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}$$

$$\leq \gamma_\nu \cdot \left(\int_{A}^{+\infty} \left|D_{*A}^{\nu+1} f(rw)\right|^{q} \cdot r^{N-1} dr\right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}. \tag{26}$$

That is we got

$$|f'(rw)| \leq \gamma_\nu \cdot \left(\int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} dr \right)^{\frac{1}{q\left(\nu+\frac{1}{p}\right)}}, \tag{27}$$

$\forall r \in [A, +\infty)$, $\quad \forall\, w \in S^{N-1}$.

Set

$$\delta_\nu := \gamma_\nu^{q\left(\nu+\frac{1}{p}\right)}. \tag{28}$$

Then by (27) we obtain

$$|f'(rw)|^{q\left(\nu+\frac{1}{p}\right)} \leq \delta_\nu \cdot \left(\int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} dr \right), \tag{29}$$

$\forall r \in [A, +\infty)$, $\quad \forall\, w \in S^{N-1}$.

Therefore

$$\int_{S^{N-1}} |f'(rw)|^{q\left(\nu+\frac{1}{p}\right)} dw \leq \delta_\nu \cdot \left(\int_{S^{N-1}} \left(\int_A^{+\infty} \left| D_{*A}^{\nu+1} f(rw) \right|^q \cdot r^{N-1} dr \right) dw \right) \tag{30}$$

$$= \delta_\nu \cdot \int_{\mathbb{R}^N - B(0,A)} \left| \frac{\partial_{*A}^{\nu+1} f(x)}{\partial r^{\nu+1}} \right|^q dx. \tag{31}$$

Consequently we derive

$$\left(\int_{S^{N-1}} \left(|f'(rw)|^{\left(\nu+\frac{1}{p}\right)} \right)^q dw \right)^{\frac{1}{q}} \leq (\delta_\nu)^{\frac{1}{q}} \cdot \left(\int_{\mathbb{R}^N - B(0,A)} \left| \frac{\partial_{*A}^{\nu+1} f(x)}{\partial r^{\nu+1}} \right|^q dx \right)^{\frac{1}{q}}$$

$$= (\delta_\nu)^{\frac{1}{q}} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{q, \mathbb{R}^N - B(0,A)}. \tag{32}$$

Therefore it holds

$$\left\| \left(f'(r\cdot) \right)^{\left(\nu+\frac{1}{p}\right)} \right\|_{q, S^{N-1}} \leq (\delta_\nu)^{\frac{1}{q}} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{q, \mathbb{R}^N - B(0,A)}, \tag{33}$$

$\forall r \in [A, +\infty)$.

Hence we get

$$\left\| \left\| \left(\frac{\partial f}{\partial r}(rw) \right)^{\left(\nu+\frac{1}{p}\right)} \right\|_{(q, S^{N-1}, w)} \right\|_{(\infty, [A,+\infty), r)} \leq (\delta_\nu)^{\frac{1}{q}} \cdot \left\| \frac{\partial_{*A}^{\nu+1} f}{\partial r^{\nu+1}} \right\|_{q, \mathbb{R}^N - B(0,A)}. \tag{34}$$

Notice that

$$(\delta_\nu)^{\frac{1}{q}} = (\gamma_\nu)^{\left(\nu+\frac{1}{p}\right)}$$

$$= (\beta_\nu)^{\left(\nu+\frac{1}{p}\right)} \cdot \left(||f||_{\infty, \mathbb{R}^N - B(0,A)} \right)^{\left(\nu-1+\frac{1}{p}\right)}$$

$$= A^{\frac{1-N}{q}} \cdot (c_\nu)^{\left(\nu+\frac{1}{p}\right)} \cdot \left(||f||_{\infty, \mathbb{R}^N - B(0,A)} \right)^{\left(\nu-1+\frac{1}{p}\right)}$$

$$= A^{\frac{1-N}{q}} \cdot \left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}} \right)^{\left(\nu-1+\frac{1}{p}\right)}.$$

$$\cdot \frac{1}{\Gamma(\nu)\cdot\left(p(\nu-1)+1\right)^{\left(\nu+\frac{1}{p}\right)/(p\nu+1)}}\cdot\left(\|f\|_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\left(\nu-1+\frac{1}{p}\right)}. \tag{35}$$

We have established the following multivariate L_p fractional Landau inequality.

Theorem 8. Let $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, with $\frac{1}{q} < \nu \leq 1$, $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$, $A > 0$ fixed, $\forall\, w \in S^{N-1}$.

Assume that $D^{\nu+1}_{*A} f(\cdot w) \in L_q([A, +\infty))$, $\forall\, w \in S^{N-1}$, $\|f\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty$ and

$$\left\|D^{\nu+1}_{*a} f(\cdot w)\right\|_{q,[a,+\infty)} \leq \left\|D^{\nu+1}_{*A} f(\cdot w)\right\|_{q,[A,+\infty)}, \forall\, a \geq A,\ \forall\, w \in S^{N-1}. \tag{36}$$

Then

$$\left\|\left\|\left(\frac{\partial f}{\partial r}(rw)\right)^{\left(\nu+\frac{1}{p}\right)}\right\|_{(q,S^{N-1},w)}\right\|_{(\infty,[A,+\infty),r)} \leq$$

$$\frac{A^{\frac{1-N}{q}}}{\Gamma(\nu)}\cdot\left(\frac{2\left(\nu+\frac{1}{p}\right)}{\nu-1+\frac{1}{p}}\right)^{\left(\nu-1+\frac{1}{p}\right)}\cdot\frac{1}{\left(p(\nu-1)+1\right)^{\frac{\left(\nu+\frac{1}{p}\right)}{(p\nu+1)}}}\cdot$$

$$\cdot\left(\|f\|_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\left(\nu-1+\frac{1}{p}\right)}\cdot\left\|\frac{\partial^{\nu+1}_{*A} f}{\partial r^{\nu+1}}\right\|_{q,\mathbb{R}^N-B(0,A)}. \tag{37}$$

It follows the $p = q = 2$ case.

Corollary 9. Let $\frac{1}{2} < \nu \leq 1$, $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$, $A > 0$ *fixed*, $\forall\, w \in S^{N-1}$.

Assume that $D^{\nu+1}_{*A} f(\cdot w) \in L_2([A, +\infty))$, $\forall\, w \in S^{N-1}$, $\|f\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty$ and

$$\left\|D^{\nu+1}_{*a} f(\cdot w)\right\|_{2,[a,+\infty)} \leq \left\|D^{\nu+1}_{*A} f(\cdot w)\right\|_{2,[A,+\infty)}, \forall\, a \geq A,\ \forall\, w \in S^{N-1}. \tag{38}$$

Then

$$\left\|\left\|\left(\frac{\partial f}{\partial r}(rw)\right)^{\left(\nu+\frac{1}{2}\right)}\right\|_{(2,S^{N-1},w)}\right\|_{(\infty,[A,+\infty),r)} \leq$$

$$\frac{A^{\frac{1-N}{2}}}{\Gamma(\nu)}\cdot\left(\frac{2\left(\nu+\frac{1}{2}\right)}{\nu-\frac{1}{2}}\right)^{\left(\nu-\frac{1}{2}\right)}\cdot\frac{1}{\left(2\nu-1\right)^{\frac{\left(\nu+\frac{1}{2}\right)}{(2\nu+1)}}}\cdot$$

$$\cdot\left(\|f\|_{\infty,\mathbb{R}^N-B(0,A)}\right)^{\left(\nu-\frac{1}{2}\right)}\cdot\left\|\frac{\partial^{\nu+1}_{*A} f}{\partial r^{\nu+1}}\right\|_{2,\mathbb{R}^N-B(0,A)}. \tag{39}$$

We finish with the $p = q = 2,\ \nu = 1$ case.

Corollary 10. Let $f(\cdot w) \in AC^2([A, b])$, $\forall b > A$, $A > 0$ *fixed* ,$\forall\, w \in S^{N-1}$. *Assume that* $f''(\cdot w) \in L_2([A, +\infty))$, $\forall\, w \in S^{N-1}$ *and* $\|f\|_{\infty,\mathbb{R}^N-B(0,A)} < \infty$.

Then

$$\left\|\left\|\left(\frac{\partial f}{\partial r}(rw)\right)^{1.5}\right\|_{(2,S^{N-1},w)}\right\|_{(\infty,[A,+\infty),r)} \leq$$

$$\left(A^{\frac{1-N}{2}}\right)\cdot\left(\sqrt{6}\right)\cdot\sqrt{\|f\|_{\infty,\mathbb{R}^N-B(0,A)}}\cdot\left\|\frac{\partial^2 f}{\partial r^2}\right\|_{2,\mathbb{R}^N-B(0,A)}. \tag{40}$$

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