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MAXIMAL AND MINIMAL POSITIVE SOLUTIONS FOR A NONLOCAL BOUNDARY VALUE PROBLEM OF A FRACTIONAL-ORDER DIFFERENTIAL EQUATION

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Abstract. In this paper we study the existence of positive solution for the fractional order differential equation $D^\beta u(t) + f(t,\ u(t)) = 0,\ t \in\ (0,1)\ ,\ \beta\ \in\ (1,2)$, with the nonlocal conditions $I^\gamma\ u(t)|_{t=0}\ =\ 0,\ \gamma \in\ (0,1],\ u(1)\ =\ k\,u(\eta),\ k > 0,\ \eta \in (a,\ b) \subset\ (0,\ 1)$ where f is L^1-Carathèodory. The existence of the maximal and minimal solutions are also studied.

1. Introduction

The three-point and nonlocal boundary value problems was studied by many authors (see for example [1-7], [9-10], [13] and [15] and references therein).
In [3], the author studied the existence of at least one positive solution for the three-point boundary-value problem

$$\begin{cases} D^\beta u(t)\ +\ f(t,\ u(t))\ = 0\ ,\ \beta\ \in\ (1,2),\ t\ \in\ (0,1), \\ u(0) = 0,\ u(1)\ =\ k\ u(\eta),\ 0\ <\ \eta\ < 1,\ 0\ <\ k\ \eta^{\beta-1}\ <\ 1. \end{cases}$$

where
(a) $f : [0,\ 1] \times [0,\ \infty)$ is nonnegative and continuous and either

$$(b)\ 0 \le\ \overline{\lim}_{u\to+\infty}\ \max_{t\in[0,\ 1]} \frac{f(t,\ u)}{u} < (1-k\eta^{\beta-1})\Gamma(\beta+1),\ \text{and}\ f(t,\ 0) \not\equiv 0,\ t \in (0,\ 1)$$

or

$$(c)\ \underline{\lim}_{u\to0^+}\ \min_{t\in[0,\ 1]} \frac{f(t,\ u)}{u} >\ \lambda_1,\ \overline{\lim}_{u\to+\infty}\ \max_{t\in[0,\ 1]} \frac{f(t,\ u)}{u} < \lambda_1.$$

In this work we omit the conditions (b) and (c), relax condition (a) and study, when f is L_1-Carathèodory, the existence of at least one positive solution for the nonlocal boundary value problem of fractional-order differential equation

$$D^\beta u(t)\ +\ f(t,\ u(t))\ = 0, \qquad \beta\ \in\ (1,2),\ t \in (0,1) \tag{1}$$

$$I^\gamma\ u(t)|_{t=0}\ =\ 0,\ \gamma\ \in\ (0,1], \qquad u(1)\ =\ k\ u(\eta),\ \eta\ \in\ (0,1). \tag{2}$$

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The maximal and minimal solutions of the problem (1)-(2) is studied when the function f is nondecreasing in the second argument.

2. Preliminaries

Let $C(I)$ denotes the class of continuous functions and $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function.

Definition 2.1 The fractional-order integral of the function $f \in L_1[a,b]$ of order $\beta > 0$ is defined by (see [12])

$$I_a^\beta f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s)\, ds,$$

Definition 2.2 The Riemann-Liouville fractional-order derivative of f of order $\beta \in (0,1)$ is defined as (see [11] and [12])

$$D_a^\beta f(t) = \frac{d}{dt} \int_a^t \frac{(t - s)^{-\beta}}{\Gamma(1 - \beta)} f(s)\, ds.$$

Definition 2.3 The function $f : [0,1] \times R \to R$ is called L^1-Caratheodory if
(i) $t \to f(t,x)$ is measurable for each $x \in R$,
(ii) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$,
(iii) there exists $m \in L^1[0,1]$ such that $|f| \leq m$.

3. Existence of solution

Lemma 3.1 The solution of the problem (1)-(2) can be represent by the integral equation

$$u(t) = \frac{A\, t^{\beta-1}}{\Gamma(\beta)} \left\{ \int_0^1 (1-s)^{\beta-1} f(s,u(s))\, ds - k \int_0^\eta (\eta - s)^{\beta-1} f(s,u(s))\, ds \right\} - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds. \quad (3)$$

where $A = (1 - k\eta^{\beta-1})^{-1}$.
proof. See [5].

Now we can write (see [3] lemma 2.4) equation (3) in the formula

$$u(t) = \int_0^1 G(t,\ s)\ f(t,\ u(s))\, ds. \quad (4)$$

where

$$G(t,\ s) = \begin{cases} \frac{-(1-k\eta^{\beta-1})(t-s)^{\beta-1} + t^{\beta-1}(1-s)^{\beta-1} - k\, t^{\beta-1}(\eta-s)^{\beta-1}}{(1-k\eta^{\beta-1})\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{t^{\beta-1}(1-s)^{\beta-1} - (1-k\eta^{\beta-1})(t-s)^{\beta-1}}{(1-k\eta^{\beta-1})\Gamma(\beta)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{t^{\beta-1}(1-s)^{\beta-1} - k\, t^{\beta-1}(\eta-s)^{\beta-1}}{(1-k\eta^{\beta-1})\Gamma(\beta)}, & 0 \leq t \leq s \leq \eta < 1, \\ \frac{t^{\beta-1}(1-s)^{\beta-1}}{(1-k\eta^{\beta-1})\Gamma(\beta)}, & 0 \leq t \leq s \leq 1, \eta \leq s. \end{cases}$$

Lemma 3.2 The function $G(t,\ s)$ satisfies $G(t,\ s) > 0$, for $t,\ s \in (0,\ 1)$.

Proof. See [3] lemma 2.4.

Definition 3.1 The function u is called a solution of the fractional-order functional integral equation (3), if $u \in C[0,1]$ and satisfies (3).

For the existence of the solution we have the following theorem.

Theorem 3.1 Assume that the the function f is L_1-Carathèodory. Then the nonlocal boundary value problem (1)-(2) has at least one positive continuous solution $u \in C[0,1]$.

Proof. Define a subset $Q_r^+ \subset C[0,1]$ by

$$Q_r^+ = \{u(t) > 0, \text{ for each } t \in [0,1], \ \|u\| \leq r\}, \text{ where } r = \frac{(1 + A + k\ A)||m||_{L_1}}{\Gamma(\beta)}.$$

The set Q_r^+ is nonempty, closed and convex.

Let $T : Q_r^+ \to Q_r^+$ be an operator defined by

$$Tu(t) = A\, t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - k\ A\, t^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds.$$

For $u \in Q_r^+$, it is clear that T is continuous operator, $i.e$ let $\{u_n(t)\}$ be a sequence in Q_r^+ converges to $u(t)$, $u_n(t) \to u(t)$, $\forall t \in [0,1]$, then

$$\lim_{n\to\infty} Tu_n(t) = A\ t^{\beta-1} \lim_{n\to\infty} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u_n(s))\, ds$$

$$-k\ A\, t^{\beta-1} \lim_{n\to\infty} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u_n(s))\, ds - \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u_n(s))\, ds$$

by assumptions (i) - (ii) and the Lebesgue dominated convergence Theorem we deduce that

$$\lim_{n\to\infty} (Tu_n)(t) = (Tu)(t).$$

Then T is continuous. Now, let $u \in Q_r^+$, then

$$\begin{aligned}(Tu)(t) &\leq A\, t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds + k\ A\, t^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &\leq A \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds + k\ A \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &\leq (1 + A + k\ A) \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &\leq \frac{(1 + A + k\ A)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}\, m(s)\, ds \\ &\leq \frac{(1 + A + k\ A)}{\Gamma(\beta)} \int_0^1 m(s)\, ds\end{aligned}$$

$$\leq \frac{(1 \ + \ A \ + \ k \ A)||m||_{L_1}}{\Gamma(\beta)} \ = \ r$$

Then $\{Tu(t)\}$ is uniformly bounded in Q_r^+.

In what follows we show that T is a completely continuous operator.
For $t_1, t_2 \in (0,1)$, $t_1 < t_2$ such that $|t_2 - t_1| < \delta$ we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= |A\ t_2^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\ ds - kAt_2^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
&- \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
&- At_1^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds + kAt_1^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
&+ \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\ ds| \\
&\leq |\int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)))\ ds - \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)))\ ds\ | \\
&+ A\ |t_2^{\beta-1}\ -\ t_1^{\beta}| \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s))|\ ds \\
&+ kA\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
&\leq |\int_0^{t_1} \left(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\ -\ \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right) f(s,u(s)))\ ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)))\ ds\ | \\
&+ A\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s))|\ ds \\
&+ kA\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s))|ds \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^{t_1} \left((t_2-s)^{\beta-1}\ -\ (t_1-s)^{\beta-1}\right) m(s)\ ds \\
&+ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-s)^{\beta-1} m(s)\ ds \\
&+ \frac{A}{\Gamma(\beta)}\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^1 (1-s)^{\beta-1} m(s)\ ds \\
&+ \frac{kA}{\Gamma(\beta)}\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^\eta (\eta-s)^{\beta-1} m(s) ds.
\end{aligned}$$

Hence the class of functions $\{Tu(t)\}$ is equi-continuous. By Arzela-Ascolis Theorem $\{Tu(t)\}$ is relatively compact. Since all conditions of Schauder Theorem are hold, then T has a fixed point in Q_r^+ .
Therefor the integral equation (3) has at least one positive continuous solution $u \in C(0,1)$.
Now,

$$\lim_{t\to 0} u(t) \ = \ A \lim_{t\to 0} t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds - kA \lim_{t\to 0} t^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds$$

$$- \lim_{t\to 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ f(s,u(s))) \ ds = u(0) = 0,$$

and

$$\lim_{t\to 1} u(t) \ = \ A \lim_{t\to 1} t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds - kA \lim_{t\to 1} t^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds$$

$$- \lim_{t\to 1} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ f(s,u(s))) \ ds = u(1).$$

Then the integral equation (3) has at least one positive continuous solution $u \in C[0,1]$.
To complete the proof operating on both sides of equation (3) by $I^{2-\beta}$, we get

$$I^{2-\beta} u(t) = \frac{A\ t}{\Gamma(\beta)} \left\{ \int_0^1 (1-s)^{\beta-1} \ f(s,u(s)) \ ds - k \ \int_0^\eta (\eta-s)^{\beta-1} \ f(s,u(s)) \ ds \right\} - I^2 \ f(t,u(t))$$

Differentiating the above relation twice we obtain the differential equation (1).
Operating on both sides of equation (3) by I^γ, we obtain

$$I^\gamma u(t) \ = \ A\, t^{\gamma+\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\gamma+\beta)} \ f(s,u(s))\, ds - k\, A\, t^{\beta-1} \ \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\gamma+\beta)} \ f(s,u(s))\, ds$$

$$- \int_0^t \frac{(t-s)^{\gamma+\beta-1}}{\Gamma(\gamma+\beta)} \ f(s,u(s)) \ ds$$

and let $t = 0$, we get

$$I^\gamma \ u(t)|_{t=0} \ = \ 0$$

Let $t = 1$ in equation (3) , we get

$$\begin{aligned} u(1) &= A \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - k\, A \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &- \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds. \\ &= (A-1) \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - k\, A \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &= (\frac{1}{1-k\eta^{\beta-1}} - 1) \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - k\, A \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &= (\frac{k\eta^{\beta-1}}{1-k\eta^{\beta-1}}) \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - k\, A \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &= k\{A\eta^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds - k\, A\eta^{\beta-1} \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\ &- \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds\} = k\, u(\eta). \end{aligned}$$

The proof is complete.■

4. Maximal and minimal solutions

Here we study the existence of the maximal and minimal solutions of the fractional-order integral equation (3).

Definition 4.1 Let n be a solution of the integral equation (3) in $[0, 1]$, then n is said to be a maximal solution of (3) if, for every solution u of (3) existing on $[0, 1]$, the inequality $u(t) \leq n(t), t \in [0, 1]$, holds.
A minimal solution may be define similarly by reversing the last inequality.
From Theorem 3.1 we get that the integral equation (3) has a positive solution $u \in C[0, 1]$.

Based on this criterion we can prove the following theorem.
Theorem 4.1 let f be a monotonic nondecreasing function in u. If the assumptions of Theorem 3.1 are satisfied , then there exist maximal and minimal solutions of the integral equation (3) on $[0, 1]$.
Proof. Consider the fractional-order integral equation

$$u_\epsilon(t) \;=\; \epsilon \;+\; \int_0^1 G(t,s)\; f(s,u(s))\; ds,\;\; \epsilon \;>\; 0. \tag{5}$$

In the view of Theorem 3.1, it is clear that equation (5) has at least one positive solution $u(t) \in C[0, 1]$. Now, let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$. Then, we have $u_{\epsilon_2}(0) < u_{\epsilon_1}(0)$ (from (3)-(5), we have $u_{\epsilon_2}(0) = \epsilon_2 < \epsilon_1 = u_{\epsilon_1}(0)$). We can prove

$$u_{\epsilon_2}(t) \;<\; u_{\epsilon_1}(t) \;\; \text{for all } t \;\in\; [0, 1]. \tag{6}$$

To prove conclusion (6), we assume that it is false, then there exist a t_1 such that

$$u_{\epsilon_2}(t_1) \;=\; u_{\epsilon_1}(t_1) \text{ and } u_{\epsilon_2}(t) \;<\; u_{\epsilon_1}(t) \;\; \text{for all } t \;\in\; [0, t_1).$$

Since f is monotonic nondecreasing in u, it follows that $f(t, u_{\epsilon_2}(t)) \leq f(t, u_{\epsilon_1}(t))$ and consequently, using equation (5), we obtain

$$\begin{aligned} u_{\epsilon_2}(t_1) &= \epsilon_2 + \int_0^1 G(t_1, s)\ f(s, u_{\epsilon_2}(s))\ ds \\ &< \epsilon_1 + \int_0^1 G(t_1, s)\ f(s, u_{\epsilon_1}(s))\ ds \\ &= u_{\epsilon_1}(t_1). \end{aligned}$$

Which contradict the fact that $u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1)$. Hence the inequality (6) is true. From the hypothesis, it follows as in the proof of Theorem 3.1 that the family of functions $\{u_\epsilon\}$ is relatively compact on $[0, 1]$, hence, we can extract a uniformly convergent subsequence $\{u_{\epsilon p}\}$, that is, there exists a decreasing sequence $\{\epsilon_p\}$ such that $\epsilon_p \to 0$ as $p \to \infty$ and $\lim_{p\to\infty} u_{\epsilon p}(t)$ exists uniformly in $t \in [0, 1]$, we denote this limiting value by $n(t)$.
Obviously, the uniform continuity of f and the equation

$$u_{\epsilon_p}(t) = \epsilon_p + \int_0^1 G(t, s)\ f(s, u_{\epsilon_p}(s))\ ds,\ t \in [0, 1],$$

yields n is a solution of equation (3). Finally, we show that the solution n is the maximal solution of equation (3). To achieve this goal, let u be any solution of (3) existing on the interval $[0, 1]$. Then

$$u(t) < \epsilon + \int_0^1 G(t, s)\ f(s, u(s))\ ds = u_\epsilon(t),\ t \in [0, 1].$$

Since the maximal solution is unique (see [8] and [14]), it is clear that $u_\epsilon(t)$ tends to $n(t)$ uniformly in $t \in [0, 1]$ as $\epsilon \to 0$. Which proves the existence of maximal solution to the integral equation (3). A similar argument holds for the minimal solution. ■

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