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ON THE FRACTIONAL-ORDER GAMES WITH NON-UNIFORM INTERACTION RATE AND ASYMMETRIC GAMES

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Abstract. The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only in Mathematics, but also in Physics, Engineering and Mathematical Biology (see [3]-[17] for example). In this paper we are concerned with the fractional-order differential equations describe Games with non-uniform interaction rate [21] and Asymmetric games [18]. Existence, uniqueness and stability of the solutions of these systems are studied.

1. Introduction

A complex adaptive system (CAS) with emergence consists of interacting adaptive agents, where the properties of the system as a whole do not exist for the individual elements (agents) and are not caused by external effects.
An emergent property of a CAS is a property of the system as a whole which does not exist at the individual elements (agents) level.
Typical examples are the brain, the immune system, the economy, social systems, ecology, insects swarm, etc... Therefore to understand a complex system one has to study the system as a whole and not to decompose it into its constituents. This totalistic approach is against the standard reductionist one, which tries to decompose any system to its constituents and hopes that by understanding the elements one can understand the whole system. Since this is quite difficult, mathematical and computer models may be helpful in studying such systems.
Recently [16] it became apparent that fractional equations naturally represent systems with memory. Moreover it has been proved that fractional order systems are relevant to fractal systems and systems with power law correlations [16].
Thus fractional equations naturally represent systems with memory and fractal systems consequently fractional order equations are relevant to CAS since memory and fractals are abundant in CAS systems.
Despite their importance, only few models have been done for fractional order CAS. Here we study some fractional order CAS systems in game theory and biology.

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We give the definition of fractional-order integration and fractional-order differentiation

Definition 1 The fractional integral of order $\beta \in R^+$ of the function $f(t),\ t \geq a$ is defined by ([17], [19] and [20])

$$I_a^\beta f(t) \ = \ \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\ ds. \tag{1}$$

The (Caputo) fractional derivative of order $\alpha \in (n-1,n)$ of $f(t),\ t \geq a$ is defined by

$$D_a^\alpha\ f(t) \ = \ I_a^{n-\alpha}\ D^n\ f(t),\ D = \frac{d}{dt}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [8] -[15], [17], [19] and [20]).

Let $\beta, \gamma \in R^+$, $\alpha \in (0,1)$ and $L_1 = L^1[a,b]$ be the class of Lebesgue integrable functions on $[a,b]$. Then we have

(i) $I_a^\beta : L^1 \to L^1$, and if $f \in L^1$, then $I_a^\gamma\ I_a^\beta\ f(t) \ = \ I_a^{\gamma+\beta}\ f(t)$.

(ii) $\lim_{\beta\to 1} I_a^\beta\ f(t) \ = \ \int_a^t\ f(s)\ ds$ uniformly on $[a,b]$.

(iii) $\lim_{\beta\to 0} I_a^\beta\ f(t) \ = \ f(t)$ weakly on $[a,b]$.

(iv) If $f(t) \ = \ k \not= 0$, k is a constant, then $D_a^\alpha\ k \ = \ 0$.

(v) Let $\beta \in (0,1)$ if $f \in C[a,b]$, then $I_a^\beta f(t)|_{t=a} = 0$

(vi) If $f(t)$ is absolutely continuous on $[a,b]$, then

$$\lim_{\alpha\to 1}\ D_a^\alpha\ f(t) \ = \ \frac{d}{dt} f(t).$$

In this paper we study the existence, uniqueness, equilibrium points and uniform stability of the solution of the fractional-order differential equation (Games with non-uniform interaction rate) [21]

$$D^\alpha x(t) = x(t)(x(t)-1)\{\frac{ar_1 x(t) + b(1-x(t))}{r_1 x(t) + 1 - x(t)} - \frac{cx(t) + dr_2(1-x(t))}{x(t) + r_2(1-x(t))}\} \tag{3}$$

with the initial data

$$x(0) = x_o. \tag{4}$$

And the system of fractional-order differential equations (Asymmetric games) [18]

$$D^\alpha x_1(t) = x_1(t)(1-x_1(t))(c_1 x_2(t) - c_2) \tag{5}$$

$$D^\alpha x_2(t) = x_2(t)(1-x_2(t))(d_1 x_1(t) - d_2) \tag{6}$$

with the initial data

$$x_1(0) = x_{o1},\ \text{and}\ x_2(0) = x_{o2}. \tag{7}$$

If $r_1 \ = \ r_2 \ = \ 1$ in (3), then equation (3) will be the replicator equation

$D^\alpha x(t) = \ x(t)(1-x(t))(A - Bx(t)),\ A,\ B > 0,\ A \ = \ b - d$ and $B \ = \ a + d - b - c$

which has been studied in [4].

If $r_1 \ = \ r_2$ and $a + d \ = \ b + c$ in (3), then equation (3) will be the logistic equation

$$D^\alpha x(t) = \rho\ x(t)(1 - x(t)),\ \rho \ = \ b - d \ > 0$$

which has been studied in [15].

For our goal here we consider firstly the initial value problem of the system of differential equation

$$D_i^\alpha x_i(t) = f_i(x_1(t), x_2(t), \cdots, x_n(t)),\ t > 0,\ i = 1, 2 \cdots, n \text{ and } x_i(0) = x_{io}. \tag{8}$$

The existence, uniqueness and uniform stability of the solution of (8) will be studied, also we prove that the equilibrium points of problem (8) are the same as the system of differential equations

$$\frac{d}{dt}x_i(t) = f_i(x_1(t), x_2(t), \cdots, x_n(t)),\ i = 1, 2, \cdots, n.$$

which proves that the results of [3], [4] and [15] (concerning the evaluation of the equilibrium points) are true.

The initial value problem (8) is also more general than the initial value problem of the fractional-order predator-prey system [4]

$$D^\alpha x_1(t) \ = \ x_1(a_1 - b_1 x_1 - x_2),\ D^\alpha x_2(t) \ = \ x_2(-a_2 + x_1) \tag{9}$$

with the initial conditions (7) and the initial value problem of the fractional-order rabies system [4]

$$D^\alpha x_1(t) \ = \ b x_1 x_2,\ D^\alpha x_2(t) \ = \ b x_1 x_2 - c x_2 \tag{10}$$

with the initial conditions (7), where $a_1,\ a_2,\ b_1,\ b, c$ are positive constants.

2. Existence and uniqueness

Let $I = [0, T],\ T < \infty$ and $C^n(I)$ be the class of all continuous column vector functions $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ defined on I, with norm

$$\|\ X\ \|^* = \sum_{i=1}^{n} \sup_t\ e^{-Nt} |\ x_i(t)\ |,\ N > 0 \tag{11}$$

which is equivalent to the sup-norm $\|\ X\ \| \ = \ \sum_{i=1}^{n} \sup_t |\ x_i(t)\ |$. When $t > \sigma \geq 0$ we write $C^n(I_\sigma)$.

Let also $L_1^n(I)$ be the class of all Lebesgue integrable column vector functions on I with norm

$$\|\ X\ \|_1 = \sum_{i=1}^{n} \|\ e^{-Nt} x_i(t)\ \|_{L_1} \ = \ \sum_{i=1}^{n} \int_0^T \ |\ e^{-Nt}\ x_i(t)\ |\ dt$$

which is equivalent to the usual norm $\|\ X\ \|_{L_1} = \sum_{i=1}^{n} \int_0^T \ |\ x_i(t)\ |dt$.

Consider now the initial value problem (8) with the following assumptions;

(1) $f_i\ :\ D\ \rightarrow\ R^+,\ D \subset R_n^+$.

(2) $\frac{\partial}{\partial\ x_j} f_i(x_1(t), x_2(t), \cdots, x_n(t)),\ i, j = 1, 2, \cdots,\ n$ exists and bounded on D.

Condition (2) implies that the functions f_i satisfy the Lipschitz condition

$$|f_i(x_1(t), \cdots, x_n(t)) - f_i(y_1(t), \cdots, y_n(t))| < K\ \|X(t) - Y(t)\ \|_n \tag{12}$$

where

$$\|\ X(t) - Y(t)\ \|_n \ = \ \sum_{i=1}^{n} |\ x_i(t) - y_i(t)\ |,\ \text{and}\ \ K\ \geq\ |\frac{\partial}{\partial\ x_j} f_i|$$

Now we have the following theorem

Theorem 1 Let the assumptions (1)-(2) be satisfied. Then the initial value problem (8) has a unique solution $X \in C^n(I),\ X' \in C^n(I_\sigma)$ and $X' \in L_1^n(I)$.

Proof. The system (8) can be written as

$$D^\alpha X(t) \ = \ F(X(t)),\ t \in (0, T] \text{ and } X(0) = X_o \tag{13}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix}, \; F(X(t)) = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ f_2(x_1, \cdots, x_n) \\ . \\ . \\ f_n(x_1, \cdots, x_n) \end{bmatrix}, \text{ and } X_o = \begin{bmatrix} x_1(0) \\ x_2(0) \\ . \\ . \\ x_n(0) \end{bmatrix}.$$

From the properties of the fractional calculus and the problem (8) we have

$$X(t) \; = \; X_o \; + \; I^\alpha \; F(X(t)) \tag{14}$$

Now let the operator $T : C^n(I) \to C^n(I)$ be defined by

$$TX(t) \; = \; X_o \; + \; I^\alpha \; F((X(t)), \tag{15}$$

then we can obtain

$$\begin{aligned} || \, e^{-Nt}(TX(t) - TY(t)) \, ||_n \; &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, e^{-N(t-s)} \, ||e^{-Ns}(F(X(s)) - F(Y(s))||_n \, ds \\ &\leq \; n \; K \; || \; X - Y \; ||^* \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; e^{-N(t-s)} \; ds \\ &= \; || \; X - Y \; ||^* \; n \; K \int_0^t \frac{s^{\alpha-1} e^{-Ns}}{\Gamma(\alpha)} ds \; < \; \frac{nK}{N^\alpha} \; || \; X - Y \; ||^* \end{aligned}$$

from which we have

$$|| \; TX - TY \; ||^* \; < \; || \; X - Y \; ||^* \frac{nK}{N^\alpha}$$

Choose N such that $N^\alpha \; > \; nK$ we deduce that

$$|| \; TX - TY \; ||^* \; < \; ||X - Y||^*$$

and the operator T has a unique fixed point. Consequently the integral equation (14) has a unique solution $X \in C^n(I)$. Also we can deduce that ([15]) $I^\alpha F(X(t))|_{t=0} = 0$.

Now from equation (14) we formally have

$$\frac{d}{dt} X(t) = F(X_o) \; \frac{t^{\alpha-1}}{\Gamma(\alpha)} + I^\alpha \; \frac{d}{dt} F(X(t)) \tag{16}$$

where

$$\frac{d}{dt} F(X(t)) = \begin{bmatrix} \sum_{i=1}^n \; \frac{\partial}{\partial x_i} f_1(x_1, \cdots, x_n) \; \frac{d}{dt} x_i \\ \sum_{i=1}^n \; \frac{\partial}{\partial x_i} f_2(x_1, \cdots, x_n) \; \frac{d}{dt} x_i \\ . \\ . \\ \sum_{i=1}^n \; \frac{\partial}{\partial x_i} f_n(x_1, \cdots, x_n) \; \frac{d}{dt} x_i \end{bmatrix}$$

and

$$|| \; \frac{d}{dt} \; F(X(t)) \; ||_n \; \leq \; n \; K|| \; \frac{d}{dt} X(t) \; ||_n,$$

then

$$|| \; \frac{d}{dt} X(t) \; ||^* \; < \; ||F(X_o) \; ||_n \; \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)} \; + \; \frac{nK}{N^\alpha} \; || \; \frac{d}{dt} X(t) \; ||^* \; \Rightarrow$$

$$|| \; \frac{d}{dt} X(t) \; ||^* < \; \frac{1}{1 - \frac{nK}{N^\alpha}} \; || \; F(X_o) \; ||_n \; \frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}$$

from which we deduce that $X' \in C^n(I_\sigma)$.
Also from (16) we can get

$$\| \frac{d}{dt} X(t) \|_1 \ < \ \frac{\| F(X_o) \|_n}{N^\alpha} \ + \ \frac{nK}{N^\alpha} \ \| \frac{d}{dt} X(t) \|_1$$

which implies that X' exists in $\in L_1^n(I)$ and given by (16).

Now let $X(t)$ be the solution of the integral equation (14), then from (16) we have

$$D^\alpha X(t) = I^{1-\alpha} \frac{d}{dt} X(t) \ = I^{1-\alpha} F(X_o) \ \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \ I^{1-\alpha} I^\alpha \frac{d}{dt} F(X(t)) = F(X(t))$$

and

$$X(t)|_{t=0} = X_o + (I^\alpha F(X(t)))|_{t=0} \ = \ X_o.$$

which proves the equivalence between the integral equation (14) and the initial value problem (8) and completes the proof of the theorem.

3. Games with non-uniform interaction rate

Lemma 2 Equation 3 can be written in the form

$$D^\alpha x(t) \ = \ f(x(t)) \ = \frac{A_1 x(t) + A_2 x^2(t) + A_3 x^3(t) + A_4 x^4(t)}{B_o + B_1 x(t) + B_2 x^2(t)} \tag{17}$$

and the function $f(x(t))$ satisfies

$$|\frac{\partial}{\partial \ x} f(x(t))| \ < \ K \tag{18}$$

where $r_1 > 1, \ 0 < r_2 < 1,$

$$K \ = \ B_o(|A_1|+2|A_2|+3|A_3|+4|A_4|)+|B_1|(|A_2|+2|A_3|+3|A_4|)+|B_2|(|A_1|+|A_3|+2|A_4|),$$

$$A_1 = (d-b)r_2, \ A_2 = (d-a)r_1 r_2 + (c-b) + 3r_2(b-d),$$

$$A_3 = (c-a)r_1 + 2r_1 r_2(a-d) + b - c) + 3r_2(d-b),$$

$$A_4 = (a-c)r_1 + (d-a)r_1 r_2 + (c-b) + r_2(b-d),$$

$$B_o = r_2, \ , B_1 = r_1 r_2 + 1 - 2r_2, \ \text{and} \ B_2 = r_1 + r_2 - r_1 r_2 - 1.$$

Now let $D = \ \{x \in R : 0 < x < 1\}$, then we have the following theorem (corollary of Theorem 1)

Theorem 2 Let $r_1 > 1$ and $0 < r_2 < 1$, Then the initial value problem (3)-(4) has a unique solution $x \in C(I), \ x' \in C(I_\sigma)$ and $x' \in L_1(I)$.

4. Asymmetric games

Lemma 3 The initial value problem (5) -(7) can be written in the form

$$D^\alpha X(t) \ = \ F(X(t)), \ t \in (0,T] \text{ and } X(0) = X_o \tag{19}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \ F(X) \ = \begin{bmatrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{bmatrix}, \ \text{and } X_o = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

where

$$f_1(x_1, x_2) = x_1(t)(1-x_1(t))(c_1 x_2(t) - c_2), \text{ and } f_2(x_1, x_2) = x_2(t)(1-x_2(t))(d_1 x_1(t) - d_2).$$

Now let $D = \{x \in R : 0 < x < 1\}$, then

$$|\frac{\partial}{\partial x_i} f_1(x_1, x_2)| < 4|c_1| + 3|c_2| \text{ and } |\frac{\partial}{\partial x_i} f_2(x_1, x_2)| < 4|d_1| + 3|d_2|.$$

Let $K = \max\,(4|c_1| + 3|c_2|,\ 4|d_1| + 3|d_2|)$, then we have the following theorem
Theorem 3 The initial value problem (5)-(7) has a unique solution $X \in C^2(I),\ X' \in C^2(I_\sigma)$ and $X' \in L_1^2(I)$.

5. Predator-prey system

Now let $D = \{x \in R : 0 < x < A\}$, $f_1(x_1(t), x_2(t)) = x_1(a_1 - b_1 x_1 - x_2)$, $f_2(x_1(t), x_2(t)) = x_2(-a_2 + x_1)$ and $k = max\{\ cA, bA, dA, r\ \}$, then we have the following theorem (corollary of Theorem 1)
Theorem 4 The initial value problem (9) and (7) has a unique solution $x \in C(I),\ x' \in C(I_\sigma)$ and $x' \in L_1(I)$.

6. Rabies system

Let $D = \{x \in R : 0 < x < A\}$, $f_1(x_1(t), x_2(t)) = x_1(a_1 - b_1 x_1 - x_2)$, $f_2(x_1(t), x_2(t)) = x_2(-a_2 + x_1)$ and $k = max\ \{r, cA\ \}$ then we have the following theorem (corollary of Theorem 1)
Theorem 5 The initial value problem (10) and(7) has a unique solution $x \in C(I),\ x' \in C(I_\sigma)$ and $x' \in L_1(I)$.

7. Stability

Let $\alpha \in (0, 1]$. The uniform stability of the solution of the initial value problems of the non-autonomous linear systems

$$D_{t_0}^{\alpha}\ x(t) \ = \ A(t)\ x(t) \ + \ f(t),\ t > t_o and\ and\ x(t_o) \ = \ x_o$$

and

$$\frac{d}{dt} x(t) \ = \ A(t)\ \frac{d}{dt}\ I_{t_0}^{\alpha}\ x(t) \ + \ f(t),\ t > t_o \text{ and } x(t_o) \ = \ x_o$$

have been studied in [1].

Consider now the initial value problem of the nonlinear system (8).
Definition 2 The solution of the problem (8) is stable if, $\forall\ \epsilon > 0$ and $t_o > 0$, there exists $\delta(\epsilon, t_o) > 0$ such that for $t \geq t_o$

$$||\ X_o - X_o^*\ ||^* < \delta(\epsilon, t_o) \ \Rightarrow \ ||\ X(t) - X^*(t)\ ||^* < \epsilon.$$

If δ depends only on ϵ, then the solution is uniformally stable [1],
where $X^*(t)$ is the solution of the initial value problem

$$D^{\alpha} x_i(t) = f_i(x_1(t), x_2(t), \cdots, x_n(t)),\ t > 0, \text{ and } x_i(0) = x^*_{o\ i}. \tag{20}$$

Now we have the following theorem;
Theorem 6 The solution of the initial value problem (8) (consequently each of the solutions of the problems (3)-(4), (5)-(7), (9) and (7) and 10 and (7)) is uniformally stable.
Proof. Let $X(t)$ and $X^*(t)$ are the solutions of the problems (8) and (20) respectively. Then by the same way as in Theorem 1, we can get

$$||\ X(t) - X^*(t)\ ||^* \ \leq \ ||\ X_o - X_o^*\ ||_n + \frac{nK}{N^{\alpha}}||\ X(t) - X^*(t)\ ||^* \ \Rightarrow$$

$$\| X(t) - X^*(t) \|^* \leq \frac{1}{1-\frac{nK}{N^\alpha}} \| X_o - X_o^* \|_n$$

from which (by definition 2) we deduce that the solution of the problem (8) is uniformally stable and the theorem is proved.

8. Equilibrium points and local stability

Consider the initial value problem

$$\frac{d}{dt}x(t) = f(x(t)), \ t > 0 \text{ and } x(0) = x_o. \tag{21}$$

To evaluate the equilibrium points of (21) let

$$\frac{d}{dt}x(t) = 0,$$

then the equilibrium points of the problem (21) are the solutions of the algebraic equation

$$f(x_{eq}) = 0.$$

To evaluate the asymptotic stability, let

$$x(t) = x_{eq} + \varepsilon(t),$$

then

$$\frac{d}{dt}(x_{eq} + \varepsilon) = f(x_{eq} + \varepsilon)$$

which implies that

$$\frac{d}{dt}\varepsilon(t) = f(x_{eq} + \varepsilon)$$

but

$$\begin{aligned} f(x_{eq} + \varepsilon) &\simeq f(x_{eq}) + f'(x_{eq})\,\varepsilon + \cdots \Rightarrow \\ f(x_{eq} + \varepsilon) &\simeq f'(x_{eq})\,\varepsilon \end{aligned}$$

where $f(x_{eq}) = 0$, then

$$\frac{d}{dt}\varepsilon(t) = f'(x_{eq})\,\varepsilon(t), \ t > 0, \text{ and } \varepsilon(0) = x_o - x_{eq}. \tag{22}$$

Now let the solution $\varepsilon(t)$ of (22) be exists. So if $\varepsilon(t)$ is increasing, then the equilibrium point x_{eq} is unstable and if $\varepsilon(t)$ is decreasing, then the equilibrium point x_{eq} is locally asymptotically stable.

Now we have the following theorem;

Theorem 7 Let the solution $x_i \in C(I), \ i = 1, 2, \cdots, n$ of the initial value problem (8) be exist. Then the equilibrium points of the problem (8) are the solutions of the algebraic equations

$$f_i(x_1(t), x_2(t), \cdots, x_n(t)) = 0$$

Proof. Consider the differential equation

$$D^\alpha x_i(t) = f_i(x_1(t), x_2(t), \cdots, x_n(t)), \ i = 1, 2, \cdots, n.$$

Then from the properties of the fractional integration we have

$$I^{1-\alpha}\frac{d}{dt}x_i(t) = f_i(x_1(t), x_2(t), \cdots, x_n(t)) \Rightarrow I\frac{d}{dt}x(t) = I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) \Rightarrow$$

$$\frac{d}{dt}x_i(t) = \frac{d}{dt}I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)).$$

Also (note that $I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t))|_{t=0} = 0$) we have

$$\frac{d}{dt}x_i(t) = \frac{d}{dt}I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) \Rightarrow$$

$$I^{1-\alpha}\frac{d}{dt}x_i(t) = I^{1-\alpha}\frac{d}{dt}I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) =$$

$$= \frac{d}{dt}I^{1-\alpha+\alpha} f_i(x_1(t), x_2(t), \cdots, x_n(t)) = f_i(x_1(t), x_2(t), \cdots, x_n(t)).$$

Then we deduce that the two differential equations

$D^\alpha x_i(t) = f_i(x_1(t), x_2(t), \cdots, x_n(t))$ and $\frac{d}{dt}x_i(t) = \frac{d}{dt}I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t))$

are equivalent.

So the equilibrium points of the problem (8) are the solutions of the algebraic equations

$$\frac{d}{dt}x_i(t) = \frac{d}{dt}I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) = 0 \Rightarrow I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) = constant = C.$$

But

$$0 = I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t))|_{t=0} = C|_{t=0} \Rightarrow C = 0 \Rightarrow I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) = 0 \Rightarrow$$

$$I f_i(x_1(t), x_2(t), \cdots, x_n(t)) = I^{1-\alpha} I^\alpha f_i(x_1(t), x_2(t), \cdots, x_n(t)) = 0 \Rightarrow$$

$$\int_0^t f_i(x_1(s), x_2(s), \cdots, x_n(s))\, ds = 0 \Rightarrow f_i(x_1(t), x_2(t), \cdots, x_n(t)) = 0.$$

and the equilibrium points of the problem (8) are the solutions of the algebraic equations

$$f_i(x_1(t), x_2(t), \cdots, x_n(t)) = 0$$

which completes the proof.

References

[1] Sh. A. Abd El-Salam and A. M. A. El-Sayed, On the stability of some fractional-order non-autonomous systems, Electronic Journal of Qualitative Theory of Differential Equations Vol. 6, 1-14, 2007.

[2] R. P. Agarwal, B. de Andrade, C. Cuevas,"On type of periodicity and ergodicity to a class of fractional order differential equations," Advances in Difference Equations, Hindawi Publ. Corp. NY, USA, Vol. 2010, Article ID 179750, 1-25.

[3] E. Ahmed, A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for the fractional replicator equation, IJMPC, 19, 7, 1-9, 2005.

[4] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models JMAA. to appear.

[5] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, On some Routh-Hurwitz condition for fractional order differential equations and thier application in Lorentz, Rosiler, Chua and Chen systems. Physics Letteres A, 358, 1, 2006.

[6] B. de Andrade,J.P.C. dos Santos, C. Cuevas, "Existence results for a fractional equations with state-dependent delay," Advances in Difference equations, Hindawi Publ. Corp. NY, USA, Volume 2011, Article ID 642013, 15 pages, doi:10.1155/2011/642013

[7] A. M. A. El-Sayed, Fractional Order Evolution Equations. Journal of Fractional Calculus, Vol. 7. 1995.

[8] A. M. A. El-Sayed, Fractional Order Diffusion-Wave Equation. International J. of Theo. Physics, Vol. 35, No. 2. 1996.
[9] A. M. A. El-Sayed, and F. M. Gaafar, Fractional calculus and some intermediate physical processes Appl. Math. and Compute. Vol. 144. 2003.
[10] A. M. A. El-Sayed, fractional differential-difference equations, Journal of Fractional Calculus, 10, 101- 106, 1996.
[11] A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Analysis: Theory, Methods and Applications., 33, 2, 181- 186, 1998.
[12] A. M. A. El-Sayed and F. M. Gaafar, Fractional order differential equations with memory and fractional-order relaxation-oscillation model (PU.M.A) Pure Math. and Appl. Vol. 12 2001.
[13] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, Comput., and Appl. Math., 23, 1, 33-54, 2004.
[14] A. M. A. El-Sayed, F. M. Gaafar and H. H. Hashem, On the maximal and minimal solutions of arbitrary orders nonlinear functional integral and differential equations. *Math Sci Res J.* 8(11) pp. 336-348. 2004.
[15] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, On the fractional-order logistic equation, J. Appl. Math. Letters Vol. 20 817-823, 2007.
[16] P. Erdi, Complexity Explained, Springer, 2008.
[17] R. Gorenflo and F.Mainardi, Fractional Calculus: Integral and Differential Equations of Fractional Order, in A. Carpinteri and F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics, Springer, 223-276, 1997, Wien.
[18] J.Hofbauer and K Sigmund, Evolutionary Games and Population Dynamics, Cambridge Univ. Press 1998.
[19] I. Podlubny and A. M. A. El-Sayed, On two definitions of fractional calculus, Solvak Academy of science-institute of experimental phys. UEF-03-96 ISBN 80-7099-252-2, 1996.
[20] I. Podlubny, Fractional differential equations, Academic Press, 1999.
[21] C.Taylor and A.M.Nowak, Evolutionary game dynamics with non-uniform interaction rate, Theor. Pop.Biol 69,243, 2006.

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